BASIC THEORY OF UNIVALENT FUNCTIONS

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1. Introduction

The modern theory of conformal mappings goes back to Göttingen in 1851, at the inaugural conference in which Riemann formulated his famous theorem on conformal mappings of simply connected domains.

Theorem 1 (Original version of the Riemann mapping theorem)**.** *Let D and G be two simply connected proper subdomains of the complex plane $\mathbb{C}$. Given $z_0 \in D$, $\xi_0 \in \partial D$, $w_0 \in G$, and $\zeta_0 \in \partial G$, there exists a unique mapping f from D onto G which is analytic and injective in D and applies z_0 into w_0 and ξ_0 into ζ_0.*

The proof that Riemann gave of this result was based on the Dirichlet principle which asserted that the problem of minimizing the integral

$$\iint_D \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy,$$

under certain conditions on the boundary has a solution. This principle was widely accepted in those days until Weierstrass observed that other similar problems in calculus of variations did not have solution: The existence of the infimum of those integral does not imply the existence of a minimizing function.

In view of this situation it became necessary to provide a new proof of Riemann's theorem. The work of very prestigious mathematicians such as C. Newmann, H. A. Schwarz, H. Poincaré, and D. Hilbert led to solve the Dirichlet problem for a very general class of domains between the end of the XIX-th century and the beginning of the XX-th century. As a consequence, the Riemann mapping theorem was proved for domains bounded by piecewise analytic curves.

The normalizations considered in the original formulation of the Riemann mapping theorem make reference to the boundaries of the domains. The boundary of a domain can be very complicated and, hence, the conditions chosen by Riemann led to unnecessary difficulties. The mapping in question is supposed to be analytic in the interior of the domain and, in general, not in the boundary. It seems more natural to consider conditions referring only to interior points. In the beginning of the XX-th century, mathematicians such as P. Koebe, C. Carathéodory, L. Bieberbach, E. Lindelöf, and P. Montel developed new methods in the theory of functions which led to simpler formulations and proofs of Riemann's theorem.

Theorem 2 (The Riemann mapping theorem)**.** *Let D be a simply connected domain in $\mathbb{C}$ with $D \neq \mathbb{C}$ and let z_0 be a point in D. Then there exists a unique mapping f*

from D onto the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ which is analytic and injective in D with $f(z_0)=0$ and $f'(z_0)>0$.

If D and G are two domains in $\mathbb{C}$ and f is a mapping which is analytic and injective in D and with $f(D)=G$, then f is said to be a conformal mapping from D onto G. Since the inverse of a conformal mapping is also a conformal mapping, the Riemann mapping theorem implies that two simply connected domains D and G in $\mathbb{C}$ with $D\neq\mathbb{C}$ and $G\neq\mathbb{C}$, are conformally equivalent.

Modern proofs of the Riemann mapping theorem use the concept of normal family. The function whose existence is insured in the statement is characterized as the element of a certain family $\mathcal{F}$ of analytic functions in D which maximizes a certain functional within $\mathcal{F}$. The existence of such a maximizing function follows from the theory of normal functions.

In addition to the obtention of a new proof of the Riemann mapping theorem, the introduction of the concept of normal family supposed a change in the way of working in geometric function theory. Until then analytic functions in a certain domain were studied individually seeking for their geometric properties. After the appearance of normal families mathematicians turned to study different classes of conformal mappings and looked for properties shared by all the elements of the family. The most studied class since then has been the class S which has a very simple definition, has the property of being compact and, hence, closed under the operation of taking limits. Furthermore, any conformal mapping from the unit disc onto a domain is related in a standard a simple way with a function in S. Consequently, any result obtained for the class S give rise to a result for arbitrary conformal mappings of the disc.

These notes are mainly devoted to study the basic properties of the class S.

2. Notation and preliminaries

2.1. **Hardy spaces.** The unit disc in the complex plane will be denoted by $\mathbb{D}$, $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, and $\mathcal{H}ol(\mathbb{D})$ will stand for the space of all analytic functions in $\mathbb{D}$.

If $a\in\mathbb{C}$ and $r>0$ the disc of center a and radius r will be denoted by $D(a,r)$, that is,

$$D(a,r)=\{z\in\mathbb{C}:|z-a|<r\}.$$

If $f\in\mathcal{H}ol(\mathbb{D})$ and $0<r<1$, we set

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it})|^p\,dt\right)^{1/p},\qquad 0<p<\infty,$$

$$I_p(r,f)=M_p(r,f)^p,\qquad 0<p<\infty,$$

$$M_\infty(r,f)=\sup_{|z|=r}|f(z)|.$$

For $0<p\leq\infty$, the Hardy space H^p is defined to be the set of all analytic functions f in the disc for which

$$\|f\|_{H^p}\stackrel{\text{def}}{=}\sup_{0<r<1}M_p(r,f)<\infty.$$

We remark that

$$H^\infty \subsetneq H^q \subsetneq H^p, \quad 0 < p < q < \infty.$$

Furthermore, a function $f \in H^p$ has a finite non-tangential limit $f(e^{i\theta})$ at almost every point $e^{i\theta} \in \partial\mathbb{D}$.

2.2. **Automorphisms of the disc.** Let $\mathbb{C}^\star = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. If D and D' are two domains in $\mathbb{C}^\star$, a conformal mapping from D onto D' is a function F which is meromorphic and injective in D and such that $F(D) = D'$. If F is a conformal mapping from D onto $\mathbb{D}'$ then F^{-1} is a conformal mapping from D' onto D.

If D is a domain in $\mathbb{C}^\star$, $\mathrm{Aut}\,(D)$ will denote the set of all conformal mappings from D onto itself. It is clear that $\mathrm{Aut}\,(D)$ is a group with the composition.

A Möbius transformation is a map $T : \mathbb{C}^\star \to \mathbb{C}^\star$ of the form

$$T(x) = \frac{az+b}{cz+d}, \quad z \in \mathbb{C}^\star$$

where a, b, c, d are complex numbers with $ad - bc \neq 0$ (it should be understood that T is defined at ∞ and at the point z_0 for which $cz_0 + d = 0$ in such a way that it becomes continuous). A Möbius transformation is a conformal mapping from $\mathbb{C}^\star$ onto itself. In fact, $\mathrm{Aut}\,(\mathbb{C}^\star)$ coincides with the set of all Möbius transformations.

A generalized circle is either a circle or a line (which can be considered as a circle through the point at infinity). Möbius transformations map generalized circles to generalized circles. From this it follows easily that the image of a disc under a Möbius transformation is either a disc, or the exterior of a disc (with the point infinity included), or a half plane.

For $a \in \mathbb{C}$ with $|a| \neq 1$, we set

$$\varphi_a(z) = \frac{z+a}{1+\overline{a}z}.$$

Then φ_a is a Möbius transformation which maps the unit circle into itself and and applies 0 into a. Also, we have that $(\varphi_a)^{-1} = \varphi_{-a}$. It turns out that the set of all Möbius transformations which map the unit circle into itself is $\{\lambda\varphi_a : |\lambda| = 1,\ |a| \neq 1\}$, while the set of all Möbius transformations which map the unit disc $\mathbb{D}$ into itself is $\{\lambda\varphi_a : |\lambda| = 1,\ |a| < 1\}$.

The Schwarz's lemma plays a basic role in the theory of analytic functions in the disc:

Theorem 3 (The Schwarz's lemma)**.** *Suppose that ω is an analytic function in the unit disc $\mathbb{D}$ with $\omega(\mathbb{D}) \subset \mathbb{D}$ and $\omega(0) = 0$. Then*

(i) $|\omega(z)| \leq |z|$, *for all* $z \in \mathbb{D}$.

(ii) $|\omega'(0)| \leq 1$.

Furthermore, if equality holds in (i) for some $z \neq 0$, or if equality holds in (ii), then ω is of the form

$$\omega(z) = \lambda z, \quad z \in \mathbb{D},$$

for a certain constant λ with $|\lambda| = 1$, in other words, ω is a rotation.

Now, if we have a function $f \in \mathcal{H}ol(\mathbb{D})$ with $f(\mathbb{D}) \subset \mathbb{D}$ and $\alpha \in \mathbb{D}$, then if $\beta = f(\alpha)$ and we apply the Schwarz lemma to the function $\omega = \varphi_{-\beta} \circ f \circ \varphi_\alpha$ we obtain the following result:

Theorem 4 (The theorem of Schwarz-Pick)**.** *Suppose that $f \in \mathcal{H}ol(\mathbb{D})$ with $f(\mathbb{D}) \subset \mathbb{D}$. Then*

(i) $\left|\frac{f(z)-f(\alpha)}{1-\overline{f(\alpha)}f(z)}\right| \leq \left|\frac{z-\alpha}{1-\overline{\alpha}z}\right|, \quad z, \alpha \in \mathbb{D}.$

(ii) $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$

Furthermore, if equality holds in (i) for some pair of points $z, \alpha \in \mathbb{D}$ with $z \neq \alpha$, or if equality holds in (ii) for some $z \in D$ then f is a Möbius transformation. If f is a Möbius transformation then equality holds in (i) for any pair of points $\alpha, z \in \mathbb{D}$ and, also, in (ii) for any $z \in \mathbb{D}$.

Remark 1. *Let us define $\rho : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ by*

$$\rho(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - \overline{z_2}z_1}\right|, \quad z_1, z_2 \in \mathbb{D}.$$

It turns out that ρ is a distance in $\mathbb{D}$ with $0 \leq \rho(z_1, z_2) < 1$, for all $z_1, z_2 \in \mathbb{D}$. The distance ρ is called the pseudohyperbolic distance in $\mathbb{D}$.

The content of the Schwarz-Pick theorem can be expressed saying that ρ in invariant under Möbius transformations of the disc into itself and that for any analytic self-map f of the disc

$$\rho\left(f(z_1), f(z_2)\right) \leq \rho\left(z_1, z_2\right), \quad z_1, z_2 \in \mathbb{D}.$$

Using the Schwarz-Pick theorem one easily obtains that any automorphism of the unit disc is a Möbius transformation. Consequently,

$$\mathrm{Aut}\,(\mathbb{D}) = \{\lambda\varphi_a : |\lambda| = 1, \quad |a| < 1\}.$$

3. Univalent functions

Definition 1. *Let D be a domain in $\mathbb{C}^\star$. A function $f : D \to \mathbb{C}^\star$ is said to be univalent in D if it is meromorphic and injective (one-to-one) in D.*

We remark that we include the assumption of meromorphy in the definition. Thus the function f is univalent in D if and only if it is analytic in D except for at most one pole and

$$f(z_1) \neq f(z_2), \quad (z_1, z_2 \in D, \quad z_1 \neq z_2).$$

Remark 2. *It is well known that if f is analytic in D and $z_0 \in D$ then f is injective in a neighborhood of z_0 if and only if $f'(z_0) \neq 0$. In other words, for f analytic in z_0 we have*

$$f'(z_0) \neq 0 \iff f \text{ is locally univalent at } z_0.$$

On the other hand, if f has a pole at z_0, then f is locally univalent at z_0 if and only if z_0 is a simple pole.

Remark 3. *If f is univalent in a domain D then it is trivially locally univalent throughout D. The converse is not true. Consider $D = \mathbb{C} \setminus \{0\}$ and $f(z) = z^2$ ($z \in D$). It is clear that f is analytic in D and locally univalent at any point of D because $f'(z_0) = 2z_0 \neq 0$ for all $z_0 \in D$. However f is not univalent in D because $f(z) = f(-z)$ for all $z \in D$.*

Remark 4. *If f is analytic in a domain D and $a \in D$, then the Jacobian of f at a is $|f'(a)|^2$. Consequently, if f is analytic and univalent in D and A is a measurable subset of D, then*

$$\text{Area } (f(A)) = \int_A |f'(z)|^2 \, dA(z).$$

If D is a domain in $\mathbb{C}^\star$ and f is a univalent function in D with $f(D) = D'$ (which is then also a domain), then we say that f is a conformal mapping from D onto D'.

Recall that the Riemann mapping theorem asserts that any simply connected proper subdomain of $\mathbb{C}$ is conformally equivalent to the unit disc. More precisely, if $D \subsetneq \mathbb{C}$ is a simply connected domain and $z_0 \in D$ then there exists a unique conformal mapping f from D onto the unit disc $\mathbb{D}$ with $f(z_0) = 0$ and $f'(0) > 0$. Therefore, statements about univalent functions in arbitrary simply connected domains can be translated into statements about univalent functions in $\mathbb{D}$. For this reason we shall study in detail univalent functions in $\mathbb{D}$.

Definition 2. *We let $\mathcal{U}$ denote the class of all functions f which are analytic and univalent in the unit disc $\mathbb{D}$.*

We also let S be the set of all functions f which are analytic and univalent in the unit disc $\mathbb{D}$ and satisfy $f(0) = 0$ and $f'(0) = 1$. That is,

$$S = \{f \in \mathcal{U} : \textit{ with } f(0) = 0 \textit{ and } f'(0) = 1\}.$$

If $f \in \mathcal{U}$ then the function g given by $g(z) = \frac{f(z)-f(0)}{f'(0)}$ ($z \in \mathbb{D}$) belongs to the class S and properties of g can be trivially translated into properties of f. Thus studying functions in the class S is sufficient to study general functions in $\mathcal{U}$.

It follows that any $f \in S$ has a Taylor expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in \mathbb{D}.$$

Examples of functions in the class S.

(i) The identity map, $f(z) = z$. We have $f(\mathbb{D}) = \mathbb{D}$.

(ii) The Möbius transformation

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \dots = \sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D},$$

which maps $\mathbb{D}$ onto the half plane $\{\operatorname{Re} z > \frac{-1}{2}\}$.

(iii) The function f defined by

$$f(z) = \frac{1}{2}\log\frac{1+z}{1-z} = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{z^{2n+1}}{2n+1}, \quad z \in \mathbb{D},$$

which maps $\mathbb{D}$ onto the strip $\{-\frac{\pi}{4} < \operatorname{Im} z < \frac{\pi}{4}\}$.

(iv) The Koebe function k defined by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots = \sum_{n=1}^{\infty} nz^n, \quad z \in \mathbb{D},$$

which maps $\mathbb{D}$ onto the domain $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq -\frac{1}{4}\}$.

The sum of two functions in the class S is not in the class S (this follows trivially from the fact that any $f \in S$ satisfies that $f'(0) = 1$). Thus, S is not a vector space. We can even assert that f is not a convex set:

Set $f(z) = \frac{z}{1-z}$, $g(z) = \frac{z}{1+iz}$ $(z \in \mathbb{D})$. It is easy to check that $f, g \in S$. We have

$$\frac{1}{2}\left(f'(z) + g'(z)\right) = \frac{1-(1-i)z}{(1-z)^2(1+iz)^2}.$$

Thus $\frac{1}{2}\left(f'(z) + g'(z)\right) = 0$ for $z = \frac{1}{1-i} = \frac{1+i}{2}$. It follows that $\frac{1}{2}(f+g)$ is not injective in $\mathbb{D}$ and, hence $\frac{1}{2}(f+g) \notin S$.

We now present several transformations which preserve S.

Theorem 5. *The class S is preserved under the following transformations:*

(i) *Rotation: If $f \in S$, $\theta \in \mathbb{R}$, and $g(z) = e^{-i\theta} f(e^{i\theta} z)$ $(z \in \mathbb{D})$, then $g \in S$.*

(ii) *Dilatation: If $f \in S$, $0 < r < 1$, and $g(z) = \frac{1}{r} f(rz)$ $(z \in \mathbb{D})$, then $g \in S$.*

(iii) *Conjugation: If $f \in S$ and $g(z) = \overline{f(\overline{z})}$ $(z \in \mathbb{D})$, then $g \in S$.*

(iv) *Disc automorphism: If $f \in \mathcal{U}$, $a \in \mathbb{D}$, and*

$$g(z) = \frac{f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)}{(1-|a|^2)f'(a)}, \quad z \in \mathbb{D},$$

then $g \in S$. This function g is usually denoted by $S_a f$.

(v) *Range transformation: If $f \in S$, $\phi : f(\mathbb{D}) \to \mathbb{C}$ is analytic and univalent in $f(\mathbb{D})$, and*

$$g(z) = \frac{\phi \circ f(z) - \phi(0)}{\phi'(0)}, \quad z \in \mathbb{D},$$

then $g \in S$.

(vi) *Omitted value transformation: If $f \in S$, $w \in \mathbb{C} \setminus f(\mathbb{D})$, and*

$$g(z) = \frac{wf(z)}{w - f(z)}, \quad z \in \mathbb{D},$$

then $g \in S$.

The proof is quite elementary. We shall include only the proof of (vi). So, suppose that $f \in S$, $w \in \mathbb{C} \setminus f(\mathbb{D})$, and

$$g(z) = \frac{wf(z)}{w - f(z)}, \quad z \in \mathbb{D}.$$

We have $g = T \circ f$, where $T(z) = \frac{wz}{w-z}$. Notice that T is a Möbius transformation whose pole is w which does not belong to $f(\mathbb{D})$. Then it follows that $g = T \circ f$ is analytic and univalent in $\mathbb{D}$. It is clear that $g(0) = 0$ and $g'(z) = \frac{w^2 f'(z)}{(w-f(z))^2}$ $(z \in \mathbb{D})$. Hence $g'(0) = 1$. Then it follows that $g \in S$.

Theorem 6 (Square-root transformation)**.** *Let $f \in S$, then there exists a function $g \in S$ such that*

$$g(z)^2 = f(z^2), \quad z \in \mathbb{D}.$$

This function g is odd and, hence, it has a Taylor series expansion of the form

$$g(z) = z + a_3 z^3 + a_5 z^5 + \cdots = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}, \quad z \in \mathbb{D}.$$

Proof. Take $f \in S$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(z \in \mathbb{D})$ (here, of course, $a_1 = 1$). We can write f in the form $f(z) = z\phi(z)$ where ϕ is analytic in $\mathbb{D}$ and $\phi(0) = 1$. Indeed,

$$\phi(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}, \quad z \in \mathbb{D}.$$

The injectivity of f implies that ϕ never vanishes in $\mathbb{D}$. Consequently, there exist a branch h of $\sqrt{\phi}$ in $\mathbb{D}$ with $h(0) = 1$, that is, h is an analytic function in $\mathbb{D}$ with $h(0) = 1$ and $h(z)^2 = \phi(z)$ for all $z \in \mathbb{D}$. The we have

$$f(z^2) = z^2\phi(z^2) = z^2 h(z^2)^2 = \left[zh(z^2)\right]^2, \quad z \in D.$$

Set $g(z) = zh(z^2)$ $(z \in \mathbb{D})$. Then $g(z)$ is a branch of $\sqrt{f(z^2)}$ and, clearly, g is an odd function. It is also clear that $g(0) = 0$, $g(z) \neq 0$ if $z \neq 0$, and

$$g'(0) = h(0) = 1.$$

Thus, it only remains to prove that g is univalent.

Suppose that $z_1, z_2 \in \mathbb{D}$ and $g(z_1) = g(z_2)$. Then $g(z_1)^2 = g(z_2)^2$ which is equivalent to saying that $f(z_1^2) = f(z_2^2)$. Since f is univalent, this implies that $z_1^2 = z_2^2$, that is, either $z_1 = z_2$, or $z_1 = -z_2$.

- If $z_1 = z_2$ we are done.
- If $z_1 = -z_2$, then using the fact that g is odd, it follows that $g(z_1) = g(z_2) = g(-z_1) = -g(z_1)$, which implies that $g(z_1) = g(z_2) = 0$ and, hence $z_1 = z_2 = 0$.

Thus in any case conclude that $z_1 = z_2$. □

With a similar argument we can prove the following result.

Theorem 7 (N-th root transformation)**.** *Let $N \geq 2$ be an integer and $f \in S$. Then there exists a function $g \in S$ such that*

$$g(z)^N = f(z^N), \quad z \in \mathbb{D}.$$

This function g satisfies

$$g(e^{\frac{2\pi i}{N}} z) = e^{\frac{2\pi i}{N}} g(z), \quad z \in \mathbb{D}.$$

The Taylor expansion of g is of the form

$$g(z) = z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \cdots = \sum_{k=0}^{\infty} a_{kN+1} z^{kN+1}, \quad z \in \mathbb{D}.$$

The image domain $g(\mathbb{D})$ has N-fold rotational symmetry, that is,

$$z \in g(\mathbb{D}) \;\Leftrightarrow\; e^{\frac{2\pi i}{N}} z \in g(\mathbb{D}).$$

Remark 5. *It is easy to prove that if a function $g \in S$ is odd then there exists $f \in S$ such that $f(z^2) = g(z)^2$ for all $z \in \mathbb{D}$. Thus the odd functions in the class S are precisely those which are obtained applying the square-root transformation to a function in S.*

More generally, if $N \geq 2$ and $g \in S$ then the following conditions are equivalent:

- $g(e^{\frac{2\pi i}{N}} z) = e^{\frac{2\pi i}{N}} g(z)$*, for all* $z \in \mathbb{D}$.
- *The image domain $g(\mathbb{D})$ has N-fold rotational symmetry.*
- *The Taylor series of g is of the form*

$$g(z) = z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \cdots = \sum_{k=0}^{\infty} a_{kN+1} z^{kN+1}, \quad z \in \mathbb{D}.$$

- *There exists $f \in S$ such that*

$$g(z)^N = f(z^N), \quad z \in \mathbb{D}.$$

Example 1. *We have $k(z^2) = \frac{z^2}{(1-z^2)^2}$ ($z \in \mathbb{D}$). Hence, applying the square-root transformation to the Koebe function we obtain the function*

$$g(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots = \sum_{n=0}^{\infty} z^{2n+1}, \quad z \in \mathbb{D}.$$

We have that $g \in S$ and $g(\mathbb{D}) = \mathbb{C} \setminus \{iy : y \in \mathbb{R} \text{ and } |y| \geq \frac{1}{2}\}$.

Similarly, applying the N-th root transformation to the Koebe fucntion, we obtain the function

$$g(z) = \frac{z}{(1-z^N)^{2/N}}, \quad z \in \mathbb{D}.$$

The function g belongs to S and the image domain $g(\mathbb{D})$ is the complement of N equally spaced radial semi-lines

$$g(\mathbb{D}) = \mathbb{C} \setminus \left(\cup_{k=0}^{N-1} \{ r e^{(2k+1)\pi i/N} : 4^{-1/N} \leq r < \infty \} \right).$$

4. The class Σ and the area theorem

Definition 3. *We let Σ to denote the class of all function F which are analytic and univalent in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$, have a simple pole at ∞, and whose Laurent expansion in Δ is of the form*

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \Delta.$$

Remark 6. *If $F \in \Sigma$ then $\mathbb{C} \setminus F(\Delta)$ is a compact and connected set.*

Remark 7. *If $f \in S$, $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ ($z \in \mathbb{D}$), then the function F defined by*

$$F(z) = f\left(\frac{1}{z}\right)^{-1} = z - a_2 + (a_2^2 - a_3)z^{-1} + \dots, \quad z \in \Delta,$$

belongs to Σ and satisfies $g(z) \neq 0$ for all z.

Conversely, if $F \in \Sigma$, $F(z) = z + b_0 + b_1 z^{-1} + \dots$ ($z \in \Delta$), and $c \in \mathbb{C} \setminus F(\Delta)$ then the function f defined by

$$f(z) = \frac{1}{F(z^{-1}) - c} = z + (c - b_0)z^2 + \dots, \quad z \in \mathbb{D},$$

belong to S.

Now we pass to state and prove the so called area theorem proved by Gronwall in 1914. It shows that the univalence of the functions in Σ has strong implications on the coefficients b_n in the Laurent expansion of F. The name "area theorem" comes from the fact that that the theorem follows from the fact that the area of a certain set is (of course) non-negative.

Theorem 8 (The Area Theorem)**.** *Suppose that $F \in \Sigma$ has the Laurent expansion*

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \Delta.$$

Then, $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

Proof. First of all, we may assume without loss of generality that $b_0 = 0$. We may also assume that $b_1 \geq 0$. Indeed, take $F \in \Sigma$ with $b_0 = 0$. For any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the function F_λ defined by $F_\lambda(z) = \lambda F\left(\frac{z}{\lambda}\right)$ also belongs to Σ and its Laurent expansion is

$$F_\lambda(z) = z + \sum_{n=1}^{\infty} \lambda^{n+1} b_n z^{-n}, \quad z \in \Delta.$$

So the quantity $\sum_{n=1}^{\infty} n|b_n|^2$ is the same for F and for F_λ. Thus, we just need to choose λ with $|\lambda| = 1$ such that $\lambda^2 b_1 \geq 0$ and work with this function F_λ in the place of F.

Thus assume that $F \in \Sigma$, $F(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$ ($z \in \Delta$), with $b_1 \geq 0$.

Set $U = \operatorname{Re} F$, $V = \operatorname{Im} F$ and $\Phi(z) = \sum_{n=2}^{\infty} b_n z^{-n}$ $(z \in \Delta)$. Thus, we have

$$F(z) = U(z) + iV(z) = z + \frac{b_1}{z} + \Phi(z), \quad z \in \Delta.$$

For $z = re^{it} \in \Delta$, we have

$$\begin{aligned} F(re^{it}) &= re^{it} + \tfrac{b_1}{r} e^{-it} + \Phi(re^{it}) \\ &= \left(r + \tfrac{b_1}{r}\right) \cos t + i \left(r - \tfrac{b_1}{r}\right) \sin t + \Phi(re^{it}) \end{aligned}$$

and then

$$\begin{aligned} U(re^{it}) &= \left(r + \tfrac{b_1}{r}\right) \cos t + \operatorname{Re}\left(\Phi(re^{it})\right) \\ V(re^{it}) &= \left(r - \tfrac{b_1}{r}\right) \sin t + \operatorname{Im}\left(\Phi(re^{it})\right). \end{aligned}$$

Set $A_r = r + \frac{b_1}{r}$, $B_r = r - \frac{b_1}{r}$ for $r > 1$ and big enough (say $r \geq r_0$) so that $B_r > 0$. We have

$$\begin{aligned} \tfrac{U(re^{it})^2}{A_r^2} + \tfrac{V(re^{it})^2}{B_r^2} &= \left(\cos t + \tfrac{\operatorname{Re}\left(\Phi(re^{it})\right)}{A_r}\right)^2 + \left(\sin t + \tfrac{\operatorname{Im}\left(\Phi(re^{it})\right)}{B_r}\right)^2 \\ &= 1 + 2\cos t \, \tfrac{\operatorname{Re}\left(\Phi(re^{it})\right)}{A_r} + 2\sin t \, \tfrac{\operatorname{Im}\left(\Phi(re^{it})\right)}{B_r} + \left(\tfrac{\operatorname{Re}\left(\Phi(re^{it})\right)}{A_r}\right)^2 + \left(\tfrac{\operatorname{Im}\left(\Phi(re^{it})\right)}{B_r}\right)^2. \end{aligned}$$

Bearing in mind the definitions of Φ, A_r and B_r, it follows that there exists a positive constant M such that

$$\frac{U(re^{it})^2}{A_r^2} + \frac{V(re^{it})^2}{B_r^2} \leq 1 + \frac{M}{r^3}, \quad r \geq r_0, t \in \mathbb{R},$$

or, equivalently,

$$\frac{U(re^{it})^2}{\left(A_r\sqrt{1 + Mr^{-3}}\right)^2} + \frac{V(re^{it})^2}{\left(B_r\sqrt{1 + Mr^{-3}}\right)^2} \leq 1. \tag{1}$$

For $r \geq r_0$, set

$$C_r = \{z \in \mathbb{C} : |z| = r\}, \;\; D_r = \{z \in \mathbb{C} : |z| > r\}, \;\; G_r = \{z \in \mathbb{C} : 1 < |z| < r\}$$

and let E_r be the ellipse

$$E_r = \left\{ z = x + iy : x \in \mathbb{R}, y \in \mathbb{R}, \frac{x^2}{\left(A_r\sqrt{1 + Mr^{-3}}\right)^2} + \frac{y^2}{\left(B_r\sqrt{1 + Mr^{-3}}\right)^2} = 1 \right\}.$$

With this notation, the inequality (1) simply says that $F(C_r)$ is contained in $I(E_r)$, the domain interior to the ellipse E_r. Also, using the univalence of F, we deduce that $F(D_r)$ is $E(F(C_r))$, the domain exterior to $F(C_r)$, and, finally,

$$F(G_r) \subset I(F(C_r)) \subset I(E_r).$$

This implies that the area of the domain $F(G_r)$ is smaller than or equal to that of $I(E_r)$. This is equivalent to saying that

$$\int_{1<|z|<r} |F'(z)|^2 \, dA(z) \leq \pi A_r B_r \left(1 + \frac{M}{r^3}\right), \quad r \geq r_0. \tag{2}$$

Now, we have

$$\begin{aligned}\int_{1<|z|<r} |F'(z)|^2\, dA(z) &= \int_1^r \rho \int_0^{2\pi} |F'(\rho e^{it})|^2\, dt\, d\rho \\ &= 2\pi \int_1^r \rho \left(1+\textstyle\sum_{n=1}^\infty n^2|b_n|^2 \rho^{-2n-2}\right) d\rho \\ &= 2\pi \int_1^r \left(\rho+\textstyle\sum_{n=1}^\infty n^2|b_n|^2 \rho^{-2n-1}\right) d\rho \\ &= 2\pi \left[\left(\tfrac{r^2}{2}-\tfrac{1}{2}\right)+\textstyle\sum_{n=1}^\infty n^2|b_n|^2 \tfrac{1}{2n}\left(1-\tfrac{1}{r^{2n}}\right)\right] \\ &= \pi\left[r^2-1+\textstyle\sum_{n=1}^\infty n|b_n|^2\left(1-\tfrac{1}{r^{2n}}\right)\right].\end{aligned}$$

Using this and (2), we obtain that, for $r \geq r_0$,

$$\begin{aligned} r^2-1+\textstyle\sum_{n=1}^\infty n|b_n|^2\left(1-\tfrac{1}{r^{2n}}\right) &\leq A_r B_r\left(1+\tfrac{M}{r^3}\right) \\ =\left(r^2-\tfrac{b_1^2}{r^2}\right)\left(1+\tfrac{M}{r^3}\right) &= r^2+\tfrac{M}{r}-\tfrac{b_1^2}{r^2}-\tfrac{b_1^2 M}{r^3} \leq r^2+\tfrac{M}{r}.\end{aligned}$$

This implies that

$$\sum_{n=1}^\infty n|b_n|^2\left(1-\frac{1}{r^{2n}}\right) \leq 1+\frac{M}{2r}, \quad r \geq r_0.$$

Letting r tend to infinity, we obtain that $\sum_{n=1}^\infty n^2|b_n| \leq 1$. □

Remark 8. *A shorter proof of the area theorem could be given using Green's theorem as follows: For $r>1$, let $A(r)$ denote the area of the domain $I(F(C_r))$ (the domain interior to the Jordan curve $F(C_r)$). Using Green's theorem we have*

$$0 < A(r) = \left|\int_{-\pi}^{\pi} U(re^{i\theta})\frac{\partial V}{\partial \theta}(re^{i\theta})\, d\theta\right| = \pi\left|r^2 - \sum_{n=1}^\infty n|b_n|^2 r^{-2n}\right|, \quad 1<r<\infty.$$

Now, $r^2 - \sum_{n=1}^\infty n|b_n|^2 r^{-2n} > 0$ for r big enough. Then, by continuity, it follows that

$$r^2 - \sum_{n=1}^\infty n|b_n|^2 r^{-2n} > 0, \quad 1<r<\infty.$$

Letting r tend to 1, we obtain $\sum_{n=1}^\infty n|b_n|^2 \leq 1$.

As an immediate consequence of the area theorem we obtain the following result.

Corollary 1. *Suppose that $F \in \Sigma$ has the Laurent expansion*

$$F(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots = z + b_0 + \sum_{n=1}^\infty b_n z^{-n}, \quad z \in \Delta.$$

Then $|b_1| \leq 1$. Furthermore the equality $|b_1|=1$ holds if and only if F is of the form

$$F(z) = z + b_0 + \frac{\lambda}{z}, \quad z \in \Delta,$$

where $b_0 \in \mathbb{C}$ *and* $|\lambda| = 1$. *This is a conformal mapping from* $\Delta \cup \{\infty\}$ *onto the complement of a segment of length* 4.

5. Coefficient estimates in the class S

Using the corollary just stated, Bieberbach proved in 1916 a bound for the coefficient a_2 of a function in the class S.

Theorem 9 (Bieberbach)**.** *Suppose that* $f \in S$, $f(z) = z + a_2z^2 + a_3z^3 + \dots$ $(z \in \mathbb{D})$. *Then* $|a_2| \leq 2$. *Furthermore the equality* $|a_2| = 2$ *holds if and only if* f *is a rotation of the Koebe function.*

Proof. Take $f \in S$, $f(z) = z + a_2z^2 + a_3z^3 + \dots$ $(z \in \mathbb{D})$. Take $g \in S$ odd such that $g(z)^2 = f(z^2)$ for all z. The Taylor expansion of g is of the form $g(z) = z + c_3z^3 + c_5z^5 + \dots$ We have then

$$(z + c_3z^3 + c_5z^5 + \dots)(z + c_3z^3 + c_5z^5 + \dots) = z^2 + a_2z^4 + \dots$$

Comparing coefficients, we obtain that $c_3 = \frac{a_2}{2}$.

Set $F(z) = \frac{1}{g\left(\frac{1}{z}\right)}$ $(z \in \Delta)$. Then $F \in \Sigma$. Also it is clear that F is odd and the Laurent series expansion of F in Δ is of the form

$$F(z) = z - \frac{c_3}{z} + \dots = z - \frac{a_2/2}{z} + \dots.$$

Using Corollary 1, we deduce that $|a_2| \leq 2$.

If $|a_2| = 2$ then Corollary 1 also yields that F is of the form $F(z) = z - \frac{\lambda}{z}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then it follows that $g(z) = \frac{z}{1-\lambda z^2}$ and $f(z) = \frac{z}{(1-\lambda z)^2}$ $(z \in \mathbb{D})$. □

If we start with a function $f \in S$, $f(z) = z + a_2z^2 + a_3z^3 + \dots$, we take directly $F(z) = \frac{1}{f\left(\frac{1}{z}\right)} = z - a_2 + (a_2^2 - a_3)z^1 + \dots$ $(z \in \Delta)$ and we apply Corollary 1 to this function F we obtain:

Theorem 10. *Suppose that* $f \in S$, $f(z) = z + a_2z^2 + a_3z^3 + \dots$ $(z \in \mathbb{D})$. *Then* $|a_2^2 - a_3| \leq 1$.

Remark 9. *For the Koebe function we have* $|a_2^2 - a_3| = 1$ *but there are other functions in* S *which give equality in Theorem 10. For example, the function* $f(z) = \frac{z}{1-z^2}$.

Corollary 2. *Suppose that* f *is an odd function in* S, $f(z) = z + c_3z^3 + c_5z^5 + \dots$ $(z \in \mathbb{D})$. *Then* $|c_3| \leq 1$. *Moreover, equality holds if and only* f *is a rotation of the function* $g(z) = \frac{z}{1-z^2}$.

We can use this result to give an easy example showing that the class S is not convex (we have already observed this fact). Take

$$f(z) = k(z) \quad \text{and} \quad g(z) = -k(-z) = z - 2z^2 + 3z^3 - 4z^4 + \dots, \qquad z \in \mathbb{D}.$$

Then $f, g \in S$ (g is a rotation of the Koebe function). Now the function $h = \frac{1}{2}f + \frac{1}{2}g$ is odd and

$$h(z) = z + 3z^3 + 5z^5 + \dots.$$

Using Corollary 2, we see that h is not univalent.

Bieberbach conjectured in his paper that if $f \in S$ then the coefficient a_n of f satisfied that $|a_n| \leq n$ with equality only for rotations of the Koebe function. This conjecture was proved for $n = 3$ by Löwner in 1923 and subsequently proofs were given for $n = 4, 5, 6$. Littlewood proved in 1925 the estimate $|a_n| \leq en$ and Bazilevic (1951) proved that $\limsup_{n\to\infty} \frac{|a_n|}{n} \leq \frac{e}{2}$. Milin proved in the 1960's that $|a_n| \leq 1.243\, n$ and FitzGerald in the 1970's that $|a_n| \leq \sqrt{\frac{7}{6}}n$. Finally, Louis de Branges proved the validity of the Bieberbach conjecture for all n in 1984.

6. The Koebe 1/4-theorem

Any function f in S is an open mapping with $0 \in f(\mathbb{D})$. Hence it follows that the range of any f in S contains a disc $D(0, r_f)$ centered at 0. To start with, the radius r_f of this disc depends on f. Koebe proved the existences of positive number δ such that the disc $D(0, \delta)$ is contained $f(\mathbb{D})$ for any $f \in S$. The Koebe function shows that $\delta \leq \frac{1}{4}$. Bieberbach proved that actually one can take $\delta = \frac{1}{4}$.

Theorem 11 (The Koebe 1/4-theorem)**.** *Let $f \in S$ then the disc $D(0, \frac{1}{4})$ of center 0 and radius 1/4 is contained in the image of f. Furthermore, if there exists a point w with $|w| = \frac{1}{4}$ and $w \notin f(\mathbb{D})$, then f is a rotation of the Koebe function.*

Proof. Take $f \in S$, $f(z) = z + a_2z^2 + a_3z^3 + \dots$ $(z \in \mathbb{D})$, and suppose that $w \notin F(\mathbb{D})$. Applying the omitted value transformation, we have that the function g defined by

$$g(z) = \frac{wf(z)}{w - f(z)}, \quad z \in \mathbb{D}$$

belongs to S. Let the Taylor series expansion of g be

$$g(z) = z + b_2z^2 + b_3z^3 + \dots, \quad z \in D.$$

We have $g(z)\,(w - f(z)) = wf(z)$ $(z \in \mathbb{D})$ and, hence,

$$\big(z + b_2z^2 + b_3z^3 + \dots\big)\big(w - z - a_2z^2 - a_3z^3 - \dots\big) = wz + wa_2z^2 + wa_3z^3 + \dots.$$

Working out the product, we see that the coefficient of z^2 in the left-hand side is $b_2w - 1$. Then we must have $b_2w - 1 = wa_2$, or, equivalently, $b_2 = \frac{1}{w} + a_2$. Since f and g belong to S we have that $|a_2| \leq 2$ and $|b_2| \leq 2$. Then it follows that

$$\left|\frac{1}{w}\right| \leq \left|\frac{1}{w} + a_2\right| + |a_2| = |b_2| + |a_2| \leq 4,$$

that, is $|w| \geq \frac{1}{4}$. Furthermore, if $|w|$ were $\frac{1}{4}$ then we would have $|a_2| = 2$ and then f would be a rotation of the Koebe function. □

Remark 10. *The univalence of the functions in S is essential to the Koebe 1/4-Theorem. If for $n = 1, 2, 3, \dots$ we set*

$$f_n(z) = \frac{1}{n}\left(e^{nz} - 1\right), \quad z \in \mathbb{D}$$

then the functions f_n are analytic in $\mathbb{D}$ and satisfy $f_n(0) = 0$ and $f_n'(0) = 1$. However, the function f_n omits the value $-\frac{1}{n}$ (which may of course be chosen arbitrarily close to zero).

As a consequence of the Koebe 1/4-Theorem (and Schwarz's Lemma), for any given simply connected proper subdomain of the plane Ω we can estimate the distance of a point $w \in \Omega$ to the boundary. This quantity is important in geometric applications.

If Ω is a domain in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$ and $w \in \Omega$, we shall let $d_\Omega(w)$ denote the distance from w to the boundary of Ω,

$$d_\Omega(w) = \operatorname{dist}(w, \partial\Omega), \quad w \in \Omega.$$

Theorem 12. *If $f \in \mathcal{U}$ and $\Omega = f(\mathbb{D})$ then*

$$\frac{1}{4}(1 - |z|^2)|f'(z)| \leq d_\Omega(f(z)) \leq (1 - |z|^2)|f'(z)|. \tag{3}$$

Proof. Take $f \in \mathcal{U}$ and $a \in \mathbb{D}$. Let

$$g(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{(1 - |a|^2)f'(a)}, \quad z \in \mathbb{D}.$$

We have that $g \in S$ (see Theorem 5). By the Koebe 1/4-theorem $D(0, \frac{1}{4}) \subset \Omega$ which is equivalent to saying that $D\left(f(a), \frac{1}{4}(1 - |a|^2)|f'(a)|\right) \subset \Omega$. Hence, we have that

$$\frac{1}{4}(1 - |a|^2)|f'(z)| \leq \operatorname{dist}(f(a), \partial\Omega).$$

To prove the upper bound we shall use Schwarz's Lemma. Set $R = \operatorname{dist}(f(a), \partial\Omega)$. Consider the mappings

$$\Phi : \mathbb{D} \to D\left(f(a), R\right), \quad \text{defined by } \Phi(z) = f(a) + Rz$$

and

$$f^{-1} : D\left(f(a), R\right) \to \mathbb{D}$$

and set $\omega(z) = \varphi_{-a} \circ f^{-1} \circ \Phi(z)$ $(z \in \mathbb{D})$, where φ_{-a} is the Möbius transformation of the disc given by $\varphi_{-a}(z) = \frac{z-a}{1-\bar{a}z}$. We have that $\omega(\mathbb{D}) \subset \mathbb{D}$ and $\omega(0) = 0$. Using the Schwarz's Lemma, we deduce that $|\omega'(0)| \leq 1$. Using the chain rule we see that

$$\omega'(0) = \frac{R}{(1 - |a|^2)f'(a)}.$$

Hence it follows that $R \leq (1 - |a|^2)|f'(a)|$. □

An analytic function f in the disc is said to be a Bloch function if

$$\varrho_{\mathcal{B}}(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. It is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + \varrho_{\mathcal{B}}(f), \quad f \in \mathcal{B}.$$

There are a lot of characterizations of Bloch functions which show up naturally in many contexts. Theorem 12 implies that if $f \in \mathcal{U}$ and $\Omega = f(\mathbb{D})$ then

$$(1-|z|^2)|f'(z)| \asymp \operatorname{dist}(f(z), \partial\Omega), \quad z \in \mathbb{D}.$$

Using this and the definition of the Bloch space, we obtain the following characterization of the univalent Bloch functions.

Theorem 13. *Let $f \in \mathcal{U}$ and $\Omega = f(\mathbb{D})$. Then the following statements are equivalent:*
(i) *$f \in \mathcal{B}$.*
(ii) *$\sup_{w\in\Omega} d_\Omega(w) < \infty$.*
(iii) *Ω dos not contain discs or arbitrarily large radius.*

Remark 11. *If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, a domain $\Omega \subset \mathbb{C}$ is said to be a univalent-X-domain if every univalent function f which maps $\mathbb{D}$ into Ω must belong to X. Also, $\Omega \subset \mathbb{C}$ is said to be an X-domain if every $f \in \mathcal{H}ol(\mathbb{D})$ which maps $\mathbb{D}$ into Ω must belong to X.*

It is easy to see that Theorem 13 implies that if $\Omega \subset \mathbb{C}$ is a simply connected domain then it is a univalent-Bloch-domain if and only if Ω dos not contain discs or arbitrarily large radius. With some more work it is possible to prove that actually this property characterized the univalent-Bloch-domains. In fact, more is true: Bloch domains are those which do not contain arbitrarily large discs.

7. Distortion theorems

Take $f \in \mathcal{U}$ and $a \in \mathbb{D}$. Applying the disc automorphism transformation S_a to f, we see that the function g defined by

$$g(z) = S_a f(z) = \frac{f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)}{(1-|a|^2)f'(a)}, \quad z \in \mathbb{D},$$

belongs to the class S. Then Bieberbach's theorem implies that $|g''(0)| \le 4$. We have

$$g'(z) = \frac{f'\left(\frac{z+a}{1+\overline{a}z}\right)}{f'(a)(1+\overline{a}z)^2}$$

$$g''(z) = \frac{(1-|a|^2)f''\left(\frac{z+a}{1+\overline{a}z}\right)}{(1+\overline{a}z)^4 f'(a)} - 2\overline{a}\frac{f'\left(\frac{z+a}{1+\overline{a}z}\right)}{f'(a)(1+\overline{a}z)^3}.$$

Hence

$$g''(0) = (1-|a|^2)\frac{f''(a)}{f'(a)} - 2\overline{a}.$$

Consequently, we have proved the following result.

Proposition 1. *Let $f \in \mathcal{U}$ and $a \in \mathbb{D}$, then*

$$\left|\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{(1-|a|^2)}\right| \leq \frac{4}{1-|a|^2}. \tag{4}$$

Remark 12. *Proposition 1 implies that if $f \in \mathcal{U}$ then $\log f'$ is a Bloch function and*

$$\|\log f'\|_{\mathcal{B}} \leq 6.$$

It turns out that a certain converse of this result is true:

If $g \in \mathcal{B}$ then there exist a function $f \in \mathcal{U}$ and a constant C such that $g = C \log f'$.

Theorem 14 (Distortion Theorem)**.** *Suppose that $f \in S$ and $z \in \mathbb{D}$. Then:*

(i) $\frac{1-|z|}{(1+|z|)^3} = k'(-|z|) \ \leq\ |f'(z)| \ \leq\ \frac{1+|z|}{(1-|z|)^3} = k'(|z|)$.

(ii) $\frac{|z|}{(1+|z|)^2} = -k(-|z|) \ \leq\ |f(z)| \ \leq\ \frac{|z|}{(1-|z|)^2} = k(|z|)$.

(iii) $\frac{1-|z|}{1+|z|} = \frac{-|z|k'(-|z|)}{k(-|z|)} \ \leq\ \left|\frac{zf'(z)}{f(z)}\right| \ \leq\ |\frac{|z|k'(|z|)}{k(|z|)} = \frac{1+|z|}{1-|z|}$.

Furthermore, equality holds in any of these inequalities for some $z \neq 0$ if and only if f is a rotation of the Koebe function.

Proof of (i). Using (4), we have

$$\left|\int_{[0,z]} \left(\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{(1-|a|^2)}\right) da\right| \leq \int_{[0,z]} \left|\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{(1-|a|^2)}\right| |da|$$

$$\leq \int_{[0,z]} \frac{4}{1-|a|^2}\,|da| \ =\ \int_0^1 \frac{4|z|}{1-t^2|z|^2}\,dt \ =\ 2\log\frac{1+|z|}{1-|z|}.$$

But

$$\int_{[0,z]} \left(\frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{(1-|a|^2)}\right) da = \log f'(z) - \int_{[0,z]} \frac{2\overline{a}}{1-|a|^2}\,da$$

$$= \log f'(z) - \int_0^1 \frac{2t|z|^2}{1-t^2|z|^2}\,dt \ =\ \log f'(z) - \log\frac{1}{1-|z|^2}.$$

Consequently, we have

$$\left|\log f'(z) - \log\frac{1}{1-|z|^2}\right| \ \leq\ 2\log\frac{1+|z|}{1-|z|}$$

which implies

$$-2\log\frac{1+|z|}{1-|z|} \leq \operatorname{Re}\left(\log f'(z) - \log\frac{1}{1-|z|^2}\right) \leq 2\log\frac{1+|z|}{1-|z|}$$

or, equivalently

$$\log\left(\frac{1-|z}{1+|z|}\right)^2 \ \leq\ \log|f'(z)| - \log\frac{1}{1-|z|^2} \ \leq\ \log\left(\frac{1+|z}{1-|z|}\right)^2.$$

This is equivalent to the inequalities in (i).

If $z \neq 0$ then equality in any of the two inequalities of (i) would imply

$$\left| \frac{f''(a)}{f'(a)} - \frac{2\overline{a}}{(1-|a|^2)} \right| = \frac{4}{1-|a|^2}, \quad \text{for all } a \in [0,z]$$

and, in particular, for $a = 0$. Hence $|f''(0)| = 4$ which, by Bieberbach theorem, implies that f is a rotation of the Koebe function. □

Proof of (ii). We have, using (i),

$$\begin{aligned} |f(z)| &= \left| \int_{[0,z]} f'(\xi)\, d\xi \right| \leq \int_{[0,z]} |f'(\xi)|\, |d\xi| \\ &= \int_0^1 |f'(tz)||z|\, dt \leq \int_0^1 k'(t|z|)|z|\, dt \;=\; k(|z|). \end{aligned}$$

Thus we have proved that $|f(z)| \leq k(|z|)$. If $z \neq 0$ and we had $|f(z)| = k(|z|)$ then we would have $|f'(\xi)| = k'(|\xi|)$, for all $\xi \in [0,z]$, and then the equality statement in (i) gives that f is a rotation of the Koebe function.

Notice that $0 \leq \frac{r}{(1+r)^2} < \frac{1}{4}$ whenever $0 \leq r < 1$. Thus, if $|f(z)| \geq \frac{1}{4}$ we have trivially that $\frac{|z|}{(1+|z|)^2} \leq |f(z)|$.

Suppose now that $|f(z)| < \frac{1}{4}$. By the Koebe 1/4-Theorem, the segment $[0, f(z)]$ is contained in $f(\mathbb{D})$. Let γ be the Jordan arc preimage of this segment:

$$\gamma(t) = f^{-1}\left(tf(z)\right), \quad t \in [0,1].$$

It is clear that

$$\int_\gamma |f'(\xi)||d\xi| = |f(z)| = \left| \int_\gamma f'(\xi)\, d\xi \right|.$$

Then, using (i), we see that

$$|f(z)| = \int_\gamma |f'(\xi)||d\xi| \geq \int_\gamma k'(-|\xi|)|d\xi|.$$

Take now a parametrization of γ of the form

$$\gamma(t) = r(t)e^{i\theta(t)}, \quad t \in [0,1].$$

Then we have

$$|f(z)| \geq \int_\gamma k'(-|\xi|)|d\xi| \geq \int_0^1 k'(-r(t))r'(t))dt = -k(-|z|).$$

If $z \neq 0$ and we had $|f(z)| = \frac{|z|}{(1-|z|)^2}$ then, necessarily, we would be in the case $|f(z)| < \frac{1}{4}$ and, furthermore, we would have $|f'(\xi)| = k'(-|\xi)|$ for all $\xi \in f^{-1}([0,z])$. Then the equality statement of (i) would imply that f is a rotation of the Koebe function. □

Proof of (iii). Take $a \in \mathbb{D}$ and $g(z) = S_a f(z)$. Then $g \in S$ and

$$|g(-a)| = \left| \frac{f(a)}{(1-|a|^2)|f'(a)} \right|.$$

Use that $g \in S$ and apply (ii) to obtain

$$\frac{|a|}{(1+|a|)^2} \le |g(-a)| = \left| \frac{f(a)}{(1-|a|^2)|f'(a)|} \right| \le \frac{|a|}{(1-|a|)^2}, \quad a \in \mathbb{D}.$$

This is equivalent to (iii). The statement about equality follows from that in (ii). □

As an immediate consequence of (ii) we obtain that S is a normal family. Even more, using Hurwitz's theorem we can assert that it is a compact family (with respect to the topology of uniform convergence in compact subsets).

Also, using (ii) we can deduce that if $f \in S$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then $|a_n| \le en^2$, for all n. To see this use that

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz, \quad 0 < r < 1. \tag{5}$$

This and part (ii) of the distortion theorem give

$$|a_n| \le \frac{1}{2\pi} 2\pi r \frac{r}{(1-r)^2} \frac{1}{r^{n+1}} = \frac{1}{(1-r)^2 r^{n-1}}, \quad 0 < r < 1.$$

Taking $r = 1 - \frac{1}{n}$ we obtain $|a_n| \le en^2$.

8. Other conjectures

W. Hayman proved in the 1950's that if $f \in S$ then

$$\alpha(f) = \lim_{r \to 1} (1-r)^2 M_\infty(r, f)$$

exists and $0 \le \alpha(f) \le 1$. Furthermore $\alpha(f) = 1$ if and only if f is a rotation of the Koebe function. The number $\alpha(f)$ is called the Hayman index of f.

Hayman also proved that if $f(z) = \sum_{n=1}^{\infty} a_n z^n \in S$, then the limit $\lim_{n\to\infty} \frac{|a_n|}{n}$ exists and coincides with $\alpha(f)$.

These two facts imply that given $f(z) = \sum_{n=1}^{\infty} a_n z^n \in S$, there exist $n(f)$ such that

$$n \ge n(f) \;\Rightarrow\; |a_n| \le n.$$

For $n = 1, 2, 3, \dots$ set

$$A_n = \sup\{|a_n| : a_n \text{ is the } n\text{-th-Taylor coefficient of some function } f \text{ in } S\}.$$

Hayman proved that $\lambda = \lim_{n\to\infty} \frac{A_n}{n}$ exists. The asymptotic Bieberbach conjecture asserted that $\lambda = 1$. This is weaker than Bieberbach conjecture.

Other classes of univalent has been considered, among them we should mention the class S_0 which consists of those $f \in \mathcal{U}$ with $f(0) = 1$ which omit the value 0. The function f_0 given by

$$f_0(z) = \left(\frac{1+z}{1-z}\right)^2 = 1 + \sum_{n=1}^{\infty} 4nz^n, \quad z \in D,$$

belongs to S_0 and is extremal for a lot of problems within this class. Littlewood conjectured asserted that if $f(z) = \sum_{n=1}^{\infty} a_n z^n \in S_0$, then $|a_n| \leq 4n$, for all n. It turns out that Littlewood conjecture and the asymptotic Bieberbach conjecture are equivalent.

We have proved that if $f(z) = \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}$ is an odd function in S then $|c_3| \leq 1$. Payley proved that there exists an absolute constant A such that $|c_{2n+1}| \leq A$ for all n. It was conjecture that A could be taken to be 1. This is not true for any $n \geq 2$.

Robertson conjectured that $1+|c_3|^2+|c_5|^2+\ldots|c_{2n-1}|^2 \leq n$, for all n. This conjecture is stronger also than Bieberbach conjecture.

For $f \in S$ consider the branch of $\log \frac{f(z)}{z}$ which takes the value 0 at 0 and write its Taylor series in the form

$$\log \frac{f(z)}{z} = 2 \sum_{j=1}^{\infty} \gamma_j z^j.$$

The γ_j's are called the logarithmic coefficients of f. For the Koebe function we have $\gamma_j = \frac{1}{j}$ for all j. Is is easy to see that if $|\gamma_j|$ were less than $\frac{1}{j}$ for all j, then one could deduce $|a_n| \leq n$ for all n. The inequality $|\gamma_j| \leq \frac{1}{j}$, for all j, is true for starlike functions in S but not in general. We remark that the inequality

$$\sum_{j=1}^{\infty} |\gamma_j|^2 r^{2j} \leq \sum_{j=1}^{\infty} \frac{1}{j^2} r^{2j}, \quad 0 < r < 1$$

is true. This was proved by Duren and Leung in 1979 and can be written in the form

$$M_2\left(r, \log \frac{f(z)}{z}\right) \leq M_2\left(r, \log \frac{f(z)}{z}\right), \quad 0 < r \leq 1, \quad f \in S.$$

The author of this notes extended this result in 1986 showing that if $0 < p \leq 2$ then

$$M_p\left(r, \log \frac{f(z)}{z}\right) \leq M_p\left(r, \log \frac{k(z)}{z}\right), \quad 0 < r \leq 1, \quad f \in S.$$

We do not know whether or not this is also true for $2 < p < \infty$.

It was conjectured that

$$\sum_{k=1}^{n} k|\gamma_k|^2 \leq \sum_{k=1}^{n} \frac{1}{k}, \text{ for all } n. \tag{6}$$

This would imply the truth of Bieberbach conjecture but (6) was shown to be false.

Milin arrived to the conjecture

$$\sum_{m=1}^{n} \sum_{j=1}^{m} \left(j|\gamma_j|^2 - \frac{1}{j}\right) \leq 0, \quad \text{for all } n.$$

It turns out that Milin conjecture implies Robertson conjecture and that this one implies the Bieberbach conjecture.

De Branges actually proved the truth of Milin conjecture.

De Branges' original proof relied on ideas from operator theory. However it was simplified and the distinct published proofs do not show the operator theory basis.

Finally, the ingredients used in the distinct published proofs have been the Löwner method (1923), a certain inequality proved by Lebedev and Milin in 1966 relating the Taylor coefficients of a function $f \in \mathcal{H}ol(\mathbb{D})$ with those of e^f, and an inequality of Askey and Gasper (1976) concerning Jacobi polynomials or some other inequalities concerning special functions.

9. SUBORDINATION

Suppose that F is a univalent function in the unit disc $\mathbb{D}$ and $\Omega = f(\mathbb{D})$; let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f(\mathbb{D}) \subset \Omega$. Then $f = F \circ \omega$ where $\omega = F^{-1} \circ f$. Notice that ω is analytic in $\mathbb{D}$ and satisfies $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$, that is, ω is in the conditions of Schwarz's Lemma. The concept of subordination generalizes this situation.

Definition 4. *Let f an F be two analytic functions in $\mathbb{D}$. We say that f is subordinated to F, written $f \prec F$, if there exists $\omega \in \mathcal{H}ol(\mathbb{D})$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$ such that $f = F \circ \omega$.*

Bearing in mind the comment made before the definition, we have:

Proposition 2. *If F is a conformal mapping from $\mathbb{D}$ onto a domain Ω and f is an analytic function in $\mathbb{D}$ with $f(0) = F(0)$ and $f(\mathbb{D}) \subset \Omega$ then $f \prec F$.*

We have:

Proposition 3. *Suppose that $f, F \in \mathcal{H}ol(\mathbb{D})$ and $f \prec F$. Then;*

(i) $f(0) = F(0)$.
(ii) $f(\mathbb{D}) \subset F(\mathbb{D})$.
(iii) $M_\infty(r, f) \le M_\infty(r, F)$, *for all* $r \in (0, 1)$.
(iv) $f(D(0, r)) \subset F(D(0, r))$, *for all* $r \in (0, 1)$.
(v) $|f'(0)| \le |F'(0)|$.
(vi) $\sup_{|z|<r}(1 - |z|^2)|f'(z)| \le \sup_{|z|<r}(1 - |z|^2)|F'(z)$, *for all* $r \in (0, 1)$.

Furthermore, if equality holds either in (iii) for some $r > 0$, or in (v) then f is of the form $f(z) = F(\lambda z)$ ($z \in \mathbb{D}$) for a certain constant λ with $|\lambda| = 1$.

(i) and (ii) are trivial; (iii), (iv), and (v) follow easily using the Schwarz's Lemma. Finally, (vi) follows using the Schwarz-Pick Theorem. We omit the details.

Remark 13. *Suppose that f and F are analytic in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $F(z) = \sum_{n=0}^{\infty} A_n z^n$ ($z \in \mathbb{D}$), and $f \prec F$. Then we have*

$$a_0 = A_0, \quad |a_1| \le |A_1|.$$

However, for $n \ge 2$ the inequality $|a_n \le |A_n|$ does not hold in general. Example: Take $f(z) = z^n$, $F(z) = z$.

Remark 14. *Notice that (vi) implies that the Bloch space is closed under subordination, that is:*

If $F \in \mathcal{B}$ and $f \prec F$, then $f \in \mathcal{B}$.

The concept of subordination can be extended to subharmonic functions.

Definition 5. *Let u an v be two subharmonic functions in $\mathbb{D}$. We say that u is subordinated to v, written $u \prec v$, if there exists $\omega \in \mathcal{H}ol(\mathbb{D})$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$ such that $u = v \circ \omega$.*

Theorem 15. *Let Let u an v be two subharmonic functions in $\mathbb{D}$ with $u \prec v$. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{it})\,dt \;\le\; \frac{1}{2\pi}\int_{-\pi}^{\pi} v(re^{it})\,dt. \tag{7}$$

Proof. We have $u = v \circ \omega$ for a certain $\omega \in \mathcal{H}ol(\mathbb{D})$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$. Take $r \in (0,1)$ and let $\{h_n\}$ be a sequence of real valued continuous functions defined on the circle $\{|z| = r\}$ with $h_n \downarrow v$ on this circle. For each n, let u_n be the solution of the Dirichlet problem in the closed disc $\{|z| \le r\}$ with boundary values h_n, that is, $u_n : \{|z| \le r\} \to \mathbb{R}$ is continuous in $\{|z| \le r\}$, harmonic in $\{|z| < r\}$, and $u_n = h_n$ on the circle $\{|z| = r\}$. By the principle of the harmonic majorant, $v \le u_n$ in $\{|z| \le r\}$. Since $\omega(\{|z| \le r\}) \subset \{|z| \le r\}$, this implies that

$$u(re^{it}) = v(\omega(re^{it})) \le u_n(\omega(re^{it})), \quad t \in \mathbb{R}.$$

Using this and the mean value property of harmonic functions, we deduce that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{it})\,dt \;\le\; \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(\omega(re^{it}))\,dt \;=\; u_n(\omega(0)) \;=\; u_n(0) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} h_n(re^{it})\,dt.$$

Then the conclusion of the theorem follows letting n tend to infinity (using the monotone convergence theorem). □

If f is analytic in $\mathbb{D}$ and $p > 0$ then $|f|^p$ is subharmonic in $\mathbb{D}$. Consequently, we have the following result.

Proposition 4. *Let f and F be two analytic functions in $\mathbb{D}$ with $f \prec F$. Then*

$$M_p(r, f) \le M_p(r, F), \quad 0 < r < 1,$$

whenever $0 < p \le \infty$.

Remark 15. *Proposition 4 for $p = 2$ gives:*

If f and F are analytic in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $F(z) = \sum_{n=0}^{\infty} A_n z^n$ ($z \in \mathbb{D}$), and $f \prec F$, then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le \sum_{n=0}^{\infty} |A_n|^2 r^{2n}, \quad 0 \le r \le 1.$$

10. Analytic functions with positive real part in the disc

Let $\mathcal{P}$ denote the class of all $f \in \mathcal{H}ol(\mathbb{D})$ with $f(0) = 1$ and Re $(f(z)) > 0$, for all $z \in \mathbb{D}$. Set

$$P(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n, \quad z \in \mathbb{D}.$$

Then P is a conformal mapping from the unit disc onto the right-half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $P(0) = 1$. Consequently, any $f \in \mathcal{P}$ is subordinated to P:

$$f \in \mathcal{P} \;\Rightarrow\; f \prec P. \tag{8}$$

Using Proposition 3 and Proposition 4, we obtain:

Theorem 16. *Suppose that $f \in \mathcal{P}$, $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^z$ $(z \in \mathbb{D})$. Then:*

(i) $|f'(0)| = |a_1| \le 2$.
(ii) $\frac{1-|z|}{1+|z|} = P(-|z|) \le |f(z)| \le \frac{1+|z|}{1-|z|} = P(|z|), \quad z \in \mathbb{D}$.
(iii) $M_p(r, f) \le M_p(r, P), \quad 0 \le r < 1, \quad 0 < p \le \infty$.
(iv) $f \in H^p$, *for all* $p \in (0, 1)$.

(i), (iii) and the upper bound in (ii) follow directly from the fact that $f \prec P$. Notice that

$$f \in \mathcal{P} \;\Rightarrow\; \frac{1}{f} \in \mathcal{P}$$

then the lower bound in (ii) follows. Finally, (iv) follows from (iii) because $P \in H^p$ for $p < 1$.

In spite of Remark 13, we have the following result.

Theorem 17. *Let $f \in \mathcal{P}$, $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Then*

$$|a_n| \le 2, \quad n = 2, 3, 4, \ldots. \tag{9}$$

This result follows from the following one about analytic functions in $\mathbb{D}$ with image contained in a convex domain.

Theorem 18. *Let $\Omega \subsetneq \mathbb{C}$ be a convex domain, and let F be a conformal mapping from the unit disc $\mathbb{D}$ onto Ω. Let f be an analytic function in $\mathbb{D}$ with $f(0) = F(0)$ and $f(\mathbb{D}) \subset \Omega$. Say that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad F(z) = \sum_{n=0}^{\infty} A_n z^n, \quad z \in \mathbb{D}.$$

Then

$$|a_n| \le |A_1|, \quad \textit{for all } n \ge 1. \tag{10}$$

Proof. We have $f \prec F$ and, hence, $a_0 = A_0$, $|a_1| \le |A_1|$.

Take $n \ge 2$. Set $\omega = e^{2\pi i/n}$. Since Ω is convex,

$$z \in \mathbb{D} \Rightarrow \frac{f(z) + f(\omega z) + f(\omega^2 z) + \ldots f(\omega^{n-1} z)}{n} \in \Omega.$$

Then, set

$$h(z)=\frac{f(z)+f(\omega z)+f(\omega^2 z)+\dots f(\omega^{n-1}z)}{n},\quad z\in\mathbb{D}.$$

We have that f is analytic in $\mathbb{D}$, $h(0)=F(0)$ and $h(\mathbb{D})\subset\Omega$. Thus $h\prec F$.

We have

$$h(z)=a_0+\sum_{k=1}^{\infty}a_k\frac{1+\omega^k+\omega^{2k}+\dots\omega^{(n-1)k}}{n}z^k,\quad z\in\mathbb{D},$$

but $\frac{1+\omega^k+\omega^{2k}+\dots\omega^{(n-1)k}}{n}$ is equal to 1 if k is a multiple of n, and 0 otherwise. Consequently, we have

$$h(z)=a_0+\sum_{j=1}^{\infty}a_{jn}z^{jn},\quad z\in\mathbb{D}.$$

Set

$$g(z)=a_0+\sum_{j=1}^{\infty}a_{jn}z^{j},\quad z\in\mathbb{D}.$$

Then g is analytic in $\mathbb{D}$ and $h(z)=g(z^n)$ $(z\in\mathbb{D})$. Then it follows that $g(\mathbb{D})\subset\Omega$, which together with the fact that $g(0)=F(0)$, implies that $g\prec F$ and then $|a_n|=|g'(0)|\le|A_1|$. □

Remark 16. *The most usual proof of (17) makes use of the Herglotz representation of functions in the class $\mathcal{P}$.*

For a function $f\in\mathcal{H}ol(\mathbb{D})$, we have that $f\in\mathcal{P}$ if and only if f can be written in the form

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+ze^{-it}}{1-ze^{-it}}\,d\mu(t)$$

for a certain positive Borel measure μ on $[-\pi,\pi]$ with total mass equal to 2π.

11. Convex and starlike functions

We define

$$\mathcal{K}=\{f\in S: f(\mathbb{D}) \text{ is a convex domain}\}.$$

$$S^\star=\{f\in S: f(\mathbb{D}) \text{ is a domain which is starlike with respect to origin }\}.$$

For instance, the functions $\frac{z}{1-z}$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ belong to $\mathcal{K}$. The Koebe function and its square root transformation $\frac{z}{1-z^2}$ belong to $S^\star$.

Theorem 18 implies the following

Theorem 19. *Suppose that $f\in\mathcal{K}$, $f(z)=\sum_{n=1}^{\infty}a_nz^n$ $(z\in\mathbb{D})$, then $|a_n|\le 1$ for all n.*

Let us give an analytic characterization of functions in $S^\star$.

Let us notice that a domain Ω in $\mathbb{C}$ is starlike with respect to the origin means that

$$w\in\Omega\ \Rightarrow\ tw\in\Omega,\ \text{for all } t\in[0,1]. \tag{11}$$

Theorem 20. *Let f be an analytic function in the unit disc $\mathbb{D}$ with $f(0) = 0$, $f'(0) = 1$. Then $f \in S^\star$ if and only if Re $\left(\frac{zf'(z)}{f(z)}\right) > 0$ for all $z \in \mathbb{D}$. In other words,*

$$f \in S^\star \;\Leftrightarrow\; \frac{zf'(z)}{f(z)} \in \mathcal{P}.$$

Proof. Suppose that $f \in S^\star$ and set $\Omega = f(\mathbb{D})$. Using (11) we see that

$$z \in \mathbb{D} \;\Rightarrow\; tf(z) \in \Omega, \text{ for all } t \in [0, 1].$$

Then it follows that $tf \prec f$ for all $t \in [0, 1]$ and this implies that

$$\{tf(z) : |z| < r\} \subset f(\{|z| < r\}, \quad (0 < r < 1,\ 0 \le t \le 1).$$

This shows that for each $r \in (0, 1)$ the domain $f(\{|z| < r\})$ is also starlike and then a simple geometric consideration yields that $\arg\left(f(re^{i\theta})\right)$ is increasing in $0 \le \theta \le 2\pi$. Hence,

$$\frac{\partial}{\partial\theta} \arg\left(f(re^{i\theta})\right) = \text{Re}\left(re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right) \ge 0, \tag{12}$$

which, since $f(0) = 0$ and $f'(0) = 1$, gives that $\frac{zf'(z)}{f(z)} \in \mathcal{P}$.

Conversely, suppose $f(0) = 0$, $f'(0) = 1$ and $p(z) = \frac{zf'(z)}{f(z)} \in \mathcal{P}$. Then $f(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$ because otherwise p would have a pole. Take $r \in (0, 1)$. Using (12) we see that $\arg\left(f(re^{i\theta})\right)$ is incresing in $0 \le \theta \le 2\pi$ and, using the argument principle, we see that the total increase is

$$\int_0^{2\pi} \frac{\partial}{\partial\theta} \arg\left(f(re^{i\theta})\right) d\theta \;=\; \text{Re}\left(\int_0^{2\pi} re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\, d\theta\right) \;=\; \text{Re}\left(\frac{1}{i}\int_{|z|=r} \frac{f'(z)}{f(z)}\, dz\right) \;=\; 2\pi$$

Then it follows that f maps the circle $\{|z| = r\}$ in an injective way onto a Jordan analytic curve J_r whose inner domain is starlike with respect to 0. It is easy to see that this implies that f in univalent in $\{|z| < r\}$ and $f(\{|z| < r\})$ is the inner domain of the curve J_r. Since this is true for any $r \in (0, 1)$, it follows that $f \in S^\star$. □

This result implies the truth of Bieberbach conjecture for functions in $S^\star$.

Theorem 21. *Let $f \in S^\star$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then $|a_n| \le n$ for all n.*

Proof. Set $p(z) = \frac{zf'(z)}{f(z)}$ $(z \in \mathbb{D})$. Then $p \in \mathcal{P}$. Write

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

we know that $|b_n| \le 2$, for all n. Comparing the Taylor coefficients in $zf'(z) = f(z)p(z)$ we see that

$$a_n = \frac{1}{n-1}\sum_{k=1}^{n-1} b_{n-k}a_k, \quad n = 2, 3, \dots.$$

Since $|b_k| \le 2$ for all k, we have

$$|a_n| \le \frac{2}{n-1}\sum_{k=1}^{n-1}|a_k|, \qquad n = 2, 3, \dots.$$

Since $a_1 = 1$ induction shows that $|a_n| \le n$ for all n. Also, we have strict inequality unless $|a_2| = 2$, that is, unless f is a rotation of the Koebe function. □

Remark 17. *Using geometric considerations and subordination we can prove that a function $f \in \mathcal{H}ol(\mathbb{D})$ with $f(0) = 0$ and $f'(0) = 1$ belongs to $\mathcal{K}$ if and only if*

$$1 + z\frac{f''(z)}{f'(z)} \in \mathcal{P}.$$

This easily implies that

$$f \in \mathcal{K} \ \Leftrightarrow\ zf'(z) \in S^\star.$$

Remark 18. *There are some other subclasses of the class S for which the Bieberbach conjecture can be proved with elementary methods. For example, let us mention that Dieudonné proved in 1931 the validity of the Bieberbach conjecture for functions $f \in S$ with real Taylor coefficients.*

12. Estimates of the integral means

The estimate $|a_n| \le en^2$ obtained at the end of in Section 7 is far from being sharp, it is not even of the right order. We deduced it from (5) and the distortion theorem. A better result would be obtained using, instead of the estimate $|f(re^{it})| \le k(r)$, an upper bound on the integral means $M_1(r, f) = \int_{-\pi}^{\pi}|f(re^{it})|\,dt$. Notice that

$$M_1(r, k) = \frac{r}{1-r^2}. \tag{13}$$

Littlewood proved in 1925 the estimate

$$\int_{-\pi}^{\pi}|f(re^{it})|\,dt \le \frac{r}{1-r}, \quad 0 < r < 1,\ f \in S. \tag{14}$$

Using this in (5) we obtain:

Theorem 22 (Littlewood)**.** *Let $f \in S$, $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Then $|a_n| \le en$ for all n.*

A. Baernstein proved that if $f \in S$ and $0 \le r < 1$ then

$$M_1(r, f) \le M_1(r, k) = \frac{r}{1-r^2}.$$

This estimate implies that $|a_n| \le \frac{e}{2}n$, for all n.

Actually, Baernstein proved a much more stronger result. He showed that the Koebe function is extremal for a very general class of problems about integral means in the class S. Namely:

Theorem 23 (Baernstein)**.** *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then for $f \in S$ and $0 < r < 1$*

$$\int_{-\pi}^{\pi} \Phi\left(\log|f(re^{it})|\right)\,dt \le \int_{-\pi}^{\pi} \Phi\left(\log|k(re^{it})|\right)\,dt.$$

Furthermore, if equality holds for son $r \in (0,1)$ and some strictly convex function Φ, then f is a rotation of the Koebe function.

In particular, taking $p > 0$ and $\Phi(x) = e^{px}$ ($x \in \mathbb{R}$), we obtain:

$$M_p(r,f) \le M_p(r,k), \quad 0 < r < 1,\ f \in S,$$

with equality for some $r \in (0,1)$ if and only f is a rotation of the Koebe function.

We shall not prove this theorem here. We shall prove a result of Prawitz (1927) which for $p = 1$ implies Littlewood's estimate, and for general p gives an upper bound of the right order for $M_p(r,f)$.

A result of Hardy and Littlewood asserts that, for any $f \in \mathcal{H}ol(\mathbb{D})$,

$$\int_0^r M_\infty(\rho,f)^p\,d\rho \le \pi I_p(r,f).$$

An inequality in the opposite direction is not true in general, but Prawitz proved one for univalent functions.

Theorem 24 (Prawitz)**.** *Let $f \in S$ and $p > 0$. Then*

$$I_p(r,f) \le p\int_0^r M_\infty(\rho,f)^p\rho^{-1}\,d\rho. \tag{15}$$

The following result will be used to prove Prawitz's theorem.

Lemma 1. *For $f \in \mathcal{H}ol(\mathbb{D})$ and $0 < p < \infty$ we have*

$$\frac{d}{dr}\left(r\frac{d}{dr}I_p(r,f)\right) = \frac{p^2 r}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{p-2}|f'(re^{i\theta})|^2\,d\theta, \quad 0 < r < 1. \tag{16}$$

Proof. Setting $z = re^{i\theta}$, we have

$$r\frac{\partial}{\partial r}|f(z)| = |f(z)|\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right), \quad \frac{\partial}{\partial\theta}|f(z)| = -|f(z)|\mathrm{Im}\left(z\frac{f'(z)}{f(z)}\right).$$

For simplicity, write $I(r)$ for $I_p(r,f)$ and

$$u(z) = \mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right), \quad v(z) = \mathrm{Im}\left(z\frac{f'(z)}{f(z)}\right).$$

The Cauchy-Riemann equations in polar coordinates take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}. \tag{17}$$

Then we have

$$r\frac{\partial}{\partial r}(|f(z)|^p) = p|f(z)|^p u(z), \quad \frac{\partial}{\partial\theta}(|f(z)|^p) = -p|f(z)|^p v(z)$$

and, hence,

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}|f(z)|^p\right) = r\frac{\partial}{\partial r}\left(p|f(z)|^p u(z)\right) = p^2|f(z)|^p u(z)^2 + p|f(z)|^p r\frac{\partial u}{\partial r}(z)$$

$$\frac{\partial^2}{\partial\theta^2}(|f(z)|^p) = \frac{\partial}{\partial\theta}(-p|f(z)|^p v(z)) = p^2|f(z)|^p v(z)^2 - p|f(z)|^p\frac{\partial v}{\partial\theta}(z).$$

Adding up and using (17), we obtain

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}|f(z)|^p\right) + \frac{\partial^2}{\partial\theta^2}(|f(z)|^p) \;=\; p^2|f(z)|^p\left|\frac{zf'(z)}{f(z)}\right|^2.$$

Since $\int_{-\pi}^{\pi}\frac{\partial^2}{\partial\theta^2}(|f(re^{i\theta})|^p)\,d\theta = 0$, it follows that

$$r\frac{d}{\partial r}\left(r\frac{d}{\partial r}(I(r))\right) = \frac{p^2}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\left|\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\,\theta,$$

which is equivalent to (16). □

Proof of Prawitz's Theorem. Take $f\in S$. Integrating (16) we obtain

$$r\frac{d}{dr}I_p(r,f) = \frac{p^2}{2\pi}\int_{|z|<r}|f(z)|^{p-2}|f'(z)|^2\,dA(z).$$

Making the change of variable $w=f(z)$, and bearing in mind that f is injective, and using that $|f(z)|\le M_\infty(r,f)$ whenever $|z|<r$, it follows that

$$\begin{aligned} r\tfrac{d}{dr}I_p(r,f) &= \tfrac{p^2}{2\pi}\int_{f(|z|<r)}|w|^{p-2}\,dA(w) \le \tfrac{p^2}{2\pi}\int_{|w|<M_\infty(r,f)}|w|^{p-2}\,dA(w)\\ &= p^2\int_0^{M_\infty(r,f)}t^{p-1}\,dt \;=\; pM_\infty(r,f)^p \end{aligned}$$

and (15) follows by integration □

Using the estimate $M_\infty(r,f)\le\frac{r}{(1-r)^2}$, Prawitz's Theorem easily implies the following result.

Theorem 25. *Let $f\in S$. Then:*

(i) *$f\in H^p$ for all $p\in(0,\frac{1}{2})$.*
(ii) *$M_1(r,f)\le\frac{r}{1-r}$, for all $r\in(0,1)$.*
(iii) *$I_{1/2}(r,f)=\mathrm{O}\left(\log\frac{1}{1-r}\right)$, as $r\to1$.*
(iv) *If $p>\frac{1}{2}$ then $I_p(r,f)=\mathrm{O}\left(\frac{1}{(1-r)^{2p-1}}\right)$, as $r\to1$.*

Notice that (ii) is (14). The Koebe function does not belong to $H^{1/2}$, hence (i) is best possible. Also, the Koebe function shows that the estimates in (iii) and (iv) are sharp.

The derivative of the Koebe function $k'(z) = \frac{1+z}{(1-z)^3}$ belongs to H^p for all $p \in (0, \frac{1}{3})$ and it does not belong to $H^{1/3}$. We also have

$$I_{1/3}(r, k') \asymp \left(\log \frac{1}{1-r}\right), \qquad \text{as } r \to 1,$$

and

$$I_p(r, k') \asymp \left(\frac{1}{(1-r)^{3p-1}}\right), \qquad \text{as } r \to 1.$$

It is natural to ask whether the analogues for the derivatives of parts (i), (iii) and (iv) Theorem 25 hold. In general, it is interesting to estimate the growth of the integral means $I_p(r, f')$ within the class S.

Of course, using de Branges's theorem (the fact that the Bieberbach conjecture is true) it follows that

$$I_2(r, f') \leq I_2(r, k'), \quad 0 < r < 1, \quad f \in S.$$

Actually this can be extended to the integral means of order a power of two:

$$I_{2^N}(r, f') \leq I_{2^N}(r, k'), \quad 0 < r < 1, \quad f \in S, \quad N = 1, 2, 3, \dots. \tag{18}$$

Let us mention also that Leung (1979) used the results of Baernstein to prove that if $f \in S^\star$, then $I_p(r, f') \leq I_p(r, k')$ $(0 < r < 1)$, for all $p > 0$.

In the negative side we have the following:

There exists a function $f \in S$ such that f' does not belong to any of the Hardy spaces. More precisely: There exists $f \in S$ such that f' dos has radial limit almost nowhere.

One way of constructing such an f is the following: Take $F(z) = \sum_{k=0}^{\infty} z^{2^k}$ $(z \in \mathbb{D})$. This function F is a Bloch function has has radial limit almost nowhere. Since $F \in \mathcal{B}$, F is of the form $F = C \log f'$ for a certain constant C and a function $f \in S$. The function f' has radial limits almost nowhere.

Let us finish with some positive results. We start obtaining an upper bound for the integral $\int_{-\pi}^{\pi} |f(re^{i\theta})|^{p-2} |f'(re^{i\theta})|^2 \, d\theta$ which appears in the right-hand side of (16).

Proposition 5. *For any given $p \in (0, \infty)$ there exists a positive constant $C = C(p)$ such that whenever $f \in S$ and $0 < r < 1$ we have*

$$\int_{-\pi}^{\pi} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \;\leq\; C(p) \frac{M_\infty(r, f)^p}{1-r}. \tag{19}$$

The following results will be used to prove Proposition 5.

Lemma 2. *There exists an absolute constant $A > 0$ (which can be taken to be $\log 2$) such that if f is a Bloch function then*

$$\left| f(\rho e^{i\theta}) - f(re^{i\theta}) \right| \leq A \|f\|_{\mathcal{B}}, \quad r \leq \rho \leq \frac{1+r}{2}, \tag{20}$$

whenever $0 < r < 1$ and $\theta \in \mathbb{R}$.

Proof. Take $f \in \mathcal{B}$, $0 < r < 1$ and $\theta \in \mathbb{R}$. For simplicity, write $r' = \frac{1+r}{2}$. Notice that $r < r' < 1$ and $1 - r' = \frac{1-r}{2}$.

For $r \le \rho \le r'$, we have

$$\left| f(\rho e^{i\theta}) - f(re^{i\theta}) \right| = \left| \int_{[re^{i\theta}, \rho e^{i\theta}]} f'(\xi)\, d\xi \right| \le \int_r^\rho |f'(te^{i\theta})|\, dt$$
$$\le \|f\|_{\mathcal{B}} \int_r^{r'} \tfrac{1}{1-t^2}\, dt \le \|f\|_{\mathcal{B}} \log \tfrac{1-r}{1-r'} = (\log 2)\|f\|_{\mathcal{B}}.$$

□

Corollary 3. *There exist a positive constant M such that if $f \in S$, $0 < r < 1$, and $\theta \in \mathbb{R}$ then*

$$(21) \qquad \frac{1}{M} \le \left| \frac{f'(\rho e^{i\theta})}{f'(re^{i\theta})} \right| \le M \quad \textit{and} \quad \frac{1}{M} \le \left| \frac{f(\rho e^{i\theta})}{f(re^{i\theta})} \right| \le M, \quad \textit{for } r \le \rho \le \frac{1+r}{2}.$$

Proof. Using Remark 12 and part (iii) of the distortion theorem we see that there exists a positive constant A such for any function $f \in S$

$$\log f' \in \mathcal{B} \text{ and } \|\log f'\|_{\mathcal{B}} \le 6$$

and

$$\log \frac{f(z)}{z} \in \mathcal{B} \text{ and } \| \log \frac{f(z)}{z} \|_{\mathcal{B}} \le 8.$$

Then Corollary 3 follows using Lemma 2. □

Proof of Proposition 5. Using Corollary 3, we see that there exists a positive constant $A(p)$ such that for whenever $f \in S$ and $f \in S$ and $0 < r < 1$, and $\theta \in \mathbb{R}$, we have

$$|f'(re^{i\theta})|^2 |f(re^{i\theta})|^{p-2} \le A(p) |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{p-2}, \quad r \le \rho \le \frac{1+r}{2}$$

Setting $r' = \frac{1+r}{2}$, and integrating, we obtain

$$\int_r^{r'} \int_0^{2\pi} r |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} d\theta d\rho \le A(p) \int_r^{r'} \int_0^{2\pi} \rho |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{\lambda-2} d\theta d\rho,$$

for $0 < r < 1$. This implies that

$$(r' - r) r \int_0^{2\pi} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{p-2} d\theta \le A(p) \int_{|z|<r} |f(z)|^{p-2} |f'(z)|^2 dA(z) =$$
$$= A(p) \int_{f\{|z|<r'\}} |w|^{p-2} dA(w) \le A(p) \int_{|w|<M_\infty(r',f)} |w|^{p-2} dA(w) \le$$
$$\le \tilde{A}(p) M_\infty(r', f)^p \le C(p) M_\infty(r, f)^p,$$

for all $r > 0$ (to obtain the last inequality we have used (21)). Since $r' - r = \frac{1-r}{2}$, this implies (19). □

Now we can prove the following result of Feng and MacGregor (1976) which asserts that if $f \in S$ then $M_p(r, f') = \mathrm{O}\left(M_p(r, k')\right)$, at least for $p > 2/5$.

Theorem 26 (Feng-MacGregor)**.** *Suppose that $\frac{2}{5}<p<\infty$ and $f\in S$. Then*

$$I_p(r,f')=\mathrm{O}\left(\frac{1}{(1-r)^{3p-1}}\right),\qquad \textit{as } r\to 1.$$

Proof of Theorem 26. Take $f\in S$ and $\frac{1}{2}<r<1$.

Suppose first that $p\ge 2$. Using that $\left|\frac{zf'(z)}{f(z)}\right|\le\frac{1+|z|}{1-|z|}$ and Proposition 5, we see that

$$\begin{aligned}\int_{-\pi}^{\pi}|f'(re^{i\theta})|^p d\theta &= \int_{-\pi}^{\pi}|f'(re^{i\theta})|^2\,|f(re^{i\theta})|^{p-2}\,\left[\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|\right]^{p-2}d\theta\\ &\le \tfrac{1}{r^{p-2}}\,\left(\tfrac{1+r}{1-r}\right)^{p-2}\int_{-\pi}^{\pi}|f'(re^{i\theta})|^2\,|f(re^{i\theta})|^{p-2}d\theta\\ &\le \tfrac{1}{r^{p-2}}\,\tfrac{1}{(1-r)^{p-2}}\,\tfrac{M_\infty(r,f)^p}{1-r}.\end{aligned}$$

Using the distortion Theorem, this gives that $I_p(r,f')\le\frac{C_p}{(1-r)^{3p-1}}$.

Suppose now that $0<p<2$. Write $p=\alpha+\beta$ with $\alpha,\beta\ge 0$.

$$\int_{-\pi}^{\pi}|f'(re^{i\theta})|^p d\theta=\int_{-\pi}^{\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^p\,|f(re^{i\theta})|^{\alpha}\,|f(re^{i\theta})|^{\beta}d\theta. \tag{22}$$

Set $\mu=\frac{2}{p}$ and let ν be the conjugate exponent of μ, that is, $\frac{1}{\mu}+\frac{1}{\nu}=1$ or $\nu=\frac{2}{2-p}$. Using Hölder's inequality we obtain

$$\begin{aligned}&\int_{-\pi}^{\pi}\left|\tfrac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^p\,|f(re^{i\theta})|^{\alpha}\,|f(re^{i\theta})|^{\beta}d\theta \qquad (23)\\ &\le\left(\int_{-\pi}^{\pi}\left|\tfrac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2\,|f(re^{i\theta})|^{\alpha\frac{2}{p}}d\theta\right)^{\frac{p}{2}}\cdot\left(\int_{-\pi}^{\pi}|f(re^{i\theta})|^{\beta\,\frac{2}{2-p}}d\theta\right)^{\frac{2-p}{2}}\end{aligned}$$

Using Lemma 5 and the distortion theorem, we obtain

$$\begin{aligned}&\left(\int_{-\pi}^{\pi}|f'(re^{i\theta})|^2\,|f(re^{i\theta})|^{\alpha\,\frac{2}{\lambda}-2}d\theta\right)^{\frac{p}{2}} \qquad (24)\\ &\le\left(\tfrac{M_\infty(r,f)^{\alpha\frac{2}{p}}}{1-r}\right)^{\frac{p}{2}}\le\tfrac{C}{(1-r)^{\left(\frac{4\alpha}{p}+1\right)\frac{p}{2}}}=\tfrac{C}{(1-r)^{2\alpha+\frac{p}{2}}}.\end{aligned}$$

Also, assuming that $\beta\cdot\frac{2}{2-p}>\frac{1}{2}$, we have

$$\left(\int_{-\pi}^{\pi}|f(re^{i\theta})|^{\beta\,\frac{2}{2-p}}d\theta\right)^{\frac{2-p}{2}}\le C\left(\frac{1}{1-r}\right)^{(\frac{4\beta}{2-p}-1)\cdot\frac{2-p}{2}}=\frac{C}{(1-r)^{2\beta+\frac{p}{2}-1}}. \tag{25}$$

Putting together (22), (23), (24), and (25), we obtain the desired estimate

$$\int_{-\pi}^{\pi}|f'(re^{i\theta})|^{\lambda}d\theta=O\left(\frac{1}{(1-r)^{3\lambda-1}}\right).$$

under the assumption that $\beta\cdot\frac{2}{2-p}>\frac{1}{2}$ or, equivalently $\beta>\frac{2-p}{4}$. But recall that $0\le\alpha=p-\beta$. Then the condition $\beta\cdot\frac{2}{2-p}>\frac{1}{2}$ implies that $0\le\alpha=p-\beta<\frac{5p-2}{4}$. Hence this is possible only for $p>\frac{2}{5}$. □

Remark 19. *The result of Feng and MacGregor for* $2 \leq p < \infty$ *can be deduced from (18) by interpolation.*

There are a lot of publications about univalent functions. We shall simply mention 4 excellent books.

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