Lecture (7)

Time – Stepping Schemes

(Grank – Nicolson Implicit Method)

7.1 Grank – Nicolson Implicit Method

The explicit method that we have taken in lecture 5 and lecture 6 is a computationally easy method. However, it has one problem concerning the time step. The time step in explicit method should be very small because the process is acceptable only if $0 \leq k/h^2 \leq \frac{1}{2}$, i.e. $k \leq \frac{1}{2}h^2$ and h must be small to attain reliable accuracy.

In 1947, Grank – Nicolson suggested and used a method to reduce overall size of calculations and it is valid (i.e. it is convergent and stable) for all values of r. A new one of implicit method replaces the previous relationship of finite difference of explicit method: (for example, diffusion equation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \qquad replaced\ by$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left\{\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right\} \text{ giving}$$

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j} \quad (7.1)$$

The left hand side of equation (7.1) contains 3 unknowns and the right hand side contains 3 knowns for the pivotal values of u (Figure 7.1). If there are N of internal grid points along each time row then for each i=1,2,… and j=0, the equation (7.1) gives N of simultaneous equations for unknown pivotal values along the first time row in terms of the known initial and boundary conditions.

Similarly, j=1 represents N of unknown values of u along the second time row in terms of the values calculated on the first row, and so on.

Such method, where the calculation of the unknown pivotal values requires solving of a set of simultaneous equations is called an *implicit method*.

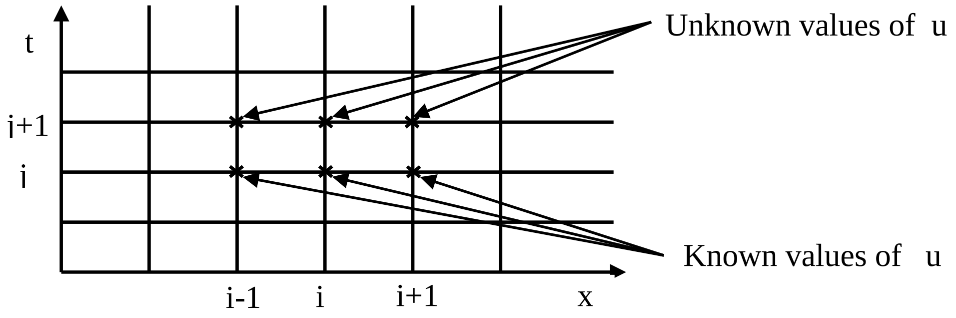


Fig. (7.1)

Example 7.1: Use the Grank-Nicolson method to calculate the numerical solution of the previous worked example, namely, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad (0 \le x \le 1)$,

where $(i)\ u = 0, x = 0\ and\ 1,\ t \ge 0,$ **(boundary conditions)**

$(ii)\ u = 2x,\ 0 \le x \le \frac{1}{2}, t = 0,\quad (iii)\ u = 2(1-x),\ \frac{1}{2} \le x \le 1\ , t = 0,$ **(initial conditions). Take h=1/10.**

Sol. Although the method is valid for all finite values of $r = k/h^2$, a large value will yield an inaccurate approximation for $\frac{\partial u}{\partial t}$. A suitable value is r=1 and has the advantage of making the coefficient of $u_{i,j}$ zero in (7.1).

The eq. (7.1) then reads as: $-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$ (7.2)

The computational molecule corresponding to equation (7.2) is shown in Fig. 7.2. For this problem, because of symmetry, $u_6 = u_4, u_7 = u_3, etc.$ See Fig (7.3).

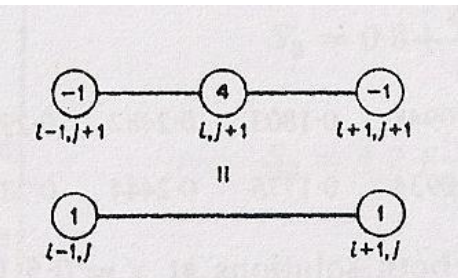


Fig (7.2)

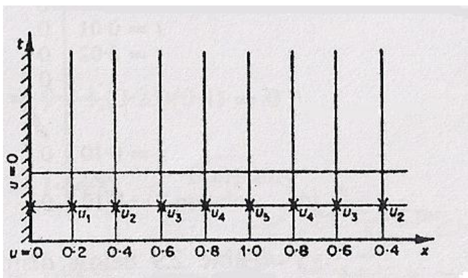


Fig (7.3)

The values of u for the first time step then satisfy

$-0 + 4u_1 - u_2 = 0 + 0.4,$

$-u_1 + 4u_2 - u_3 = 0.2 + 0.6,$

$-u_2 + 4u_3 - u_4 = 0.4 + 0.8,$

$-u_3 + 4u_4 - u_5 = 0.6 + 1.0,$

$-2u_4 + 4u_5 = 0.8 + 0.8.$

These equations are easily solved by systematic eliminations to give:

$u_1=0.1989, \quad u_2=0.3956, \quad u_3=0.5834, \quad u_4=0.7381, \quad u_5=0.7691$

Hence the equations for the pivotal values of u along the next time row are:

$-0 + 4u_1 - u_2 = 0 + 0.3956,$

$-u_1 + 4u_2 - u_3 = 0.1989 + 0.5834,$

$-u_2 + 4u_3 - u_4 = 0.3956 + 0.7381,$

$-u_3 + 4u_4 - u_5 = 0.5834 + 0.7691,$

$-2u_4 + 4u_5 = 1.4762$

The solution of these equations is given in Table (7.1). There is a comparison with the analytical solution of the partial differential equations at t=0.1 . The numerical solution here is clearly a good one.

Table (7.1)

x = 0		0.1	0.2	0.3	0.4	0.5
t=0.00	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
t=0.01	**0**	**0.1989**	**0.3956**	**0.5834**	**0.7381**	**0.7691**
t=0.02	**0**	**0.1936**	**0.3789**	**0.5400**	**0.6461**	**0.6921**
⋮						
t=0.10	**0**	**0.0948**	**0.1803**	**0.2482**	**0.1918**	**0.3069**
Analytical solution t=0.10	**0**	**0.0934**	**0.1776**	**0.2444**	**0.2873**	**0.3021**

Table (7.2) below displays both solutions at x=0.5 for various values of t. It is clear that the accuracy of this implicit method over the time range taken is about the same as for the explicit method which uses ten times as many time steps.

Table (7.2)

	Finite –difference Solution (x=0.5)	Analytical solution (x=0.5)	Difference	Percentage error
t=0.01	0.7691	0.7743	-0.0052	-0.7
t=0.02	0.6921	0.6809	+0.0112	1.6
t=0.10	0.3069	0.3021	0.0048	1.6

Example 7.1: Use Grank-Necolson method to calculate the numerical solution of the following PDE (at the first row only): $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad (\mathbf{0} \le \boldsymbol{x} \le \mathbf{1})$

where u=4.5 at x=0 and u=1.5 at x=1, t ≥ 0

u=3(1.5-x) at t=0, take h=0.2, r=1.

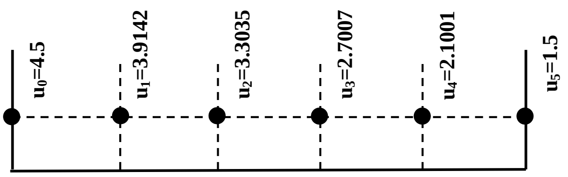
Sol. 1. We write the above equation in finite difference method as:

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j}$$

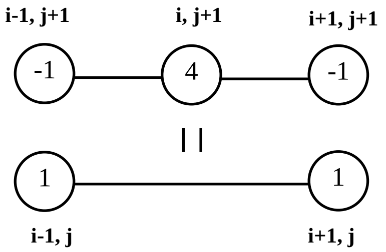
where r=k/h^2. At r=1, we get:

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

2. We draw the grid and the molecule



x	0	0.2	0.4	0.6	0.8	1
u	4.5	3.9	3.3	2.9	2.1	1.5



3. Establishing initial values

u=3(1.5-0.2)=3.9

u=3(1.5-0.4)=3.3

u=3(1.5-0.6)=2.9

u=3(1.5-0.8)=2.1

4. Set the simultaneous equations.

$-4.5+4u_1-u_2=4.5+3.3$ …(1)

$-u_1+4u_2-u_3=3.9+2.9$ …(2)

$-u_2+4u_3-u_4=3.3+2.1$ …(3)

$-u_3+4u_4-1.5=2.7+1.5$ …(4)

5. Finding u-values

From eq. 4

$4u_4=u_3+5.7 \Rightarrow u_4=0.25u_3+1.425$ …(5)

we substitute the value of u_4 in eq. 3

$-u_2+4u_3-0.25u_3-1.425=5.4 \Rightarrow - u_2+3.75u_3=6.825 \Rightarrow u_3= 0.2666u_2+1.82$ …(6)

Sub. in eq2

$-u_1+4u_2-0.2666u_2-1.82=6.8 \Rightarrow -u_1+3.7334u_2=8.62 \Rightarrow u_2=0.2678u_1+2.3088$ …(7)

Sub. in eq1

$-4.5+4u_1-0.2678u_1-2.3088=7.8 \Rightarrow -4.5+3.7322u_1=10.1088 \Rightarrow 3.7322u_1=14.6088 \Rightarrow$

$u_1=3.9142$ substituted in eq.7 $\Rightarrow u_2=0.2678*3.9142 +2.2553= 3.3035$, Sub. in eq.6

$u_3= 0.2666*3.3035+1.82 =2.7007$, Sub. in eq.5

$u_4=0.25*2.7007 +1.425=2.1001$