Lecture (8)

Consistency, Convergence, and Stability

8.1 Criteria of Optimal Numerical Solution

The numerical solution can be very close to the exact solution of the partial differential equations if many criteria meet.

First, the finite-difference analog in space and time must **converge** to its differential expression when the terms in the analog approach zero. For example, if $\Delta N/\Delta x$ is the finite difference analog of $\partial N/\partial x$, then the following convergence criterion:

$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \left\| \frac{\Delta N}{\Delta x} \right\| \qquad (8.1)$$

must be available to say that the approximation is accurate.

Second, the finite difference analog must be **consistent**. The finite-difference analog $\Delta N/\Delta x$ in equation (8.1) can be extracted from Tayler series expansion. In the expansion, the higher order terms are neglected to reduce the arithmetic burden of approximation. Difference between full Tayler series expansion and truncated approximation is the truncation error. The finite difference approximation for a derivative is *consistent* if the truncation error of the approximation approaches zero. We can say the consistence can be occurred when:

$$\lim_{\Delta x \to 0} \left\| TE\left(\frac{\Delta N}{\Delta x}\right) \right\| = 0 \qquad (8.2)$$

where TE is the truncation error of the approximation $\Delta N/\Delta x$.

Third, if the finite difference approximation is consistent, then the rate that its truncation error approaches zero depends on the order of approximation. The *order of approximation* is the order of the term of the lowest rank in the neglected Tayler series expansion in the approximation. The higher the order of approximation, the faster the truncation error approaches zero within a given spatial (or temporal) resolution. Thus with the same Δx, the high-order approximation is more accurate than the lower-order approximation. For the same truncation error, a low-order approximation requires smaller Δx than a high order approximation. As a result, the high order approximation with high Δx can has the same truncation error of the low order approximation with small Δx.

Because the high order approximation includes more terms, it requires more calculations than a low-order approximation with the same Δx . Having a high order with respect to one variable, such as space, is only useful if the order of the other variable, such as time, is also high. Otherwise, a low resolution in the time derivative will overwhelm the high accuracy in the space derivative. Thus, the optimal finite difference solution has an identical order in space and time.

Fourth, while the analogs of the individual finite differences should converge towards exact differentials, the total numerical solution of PDE should converge to the exact solution when the spatial or temporal differences decrease towards zero. If we consider $N_{e,x,t}$ is exact solution, and $N_{f,x,t}$ is a finite difference solution of PDE, then the overall convergence occurs when:

$$\lim_{\Delta x,\Delta t \to 0} \left\| N_{e,x,t} - N_{f,x,t} \right\| = 0 \qquad (8.3)$$

If the finite difference solution is nonconvergent then it is not useful.

Fifth, for the numerical method to be successful, it must be stable. **Stability** occurs when the absolute value difference between numerical and exact solutions does not grow with time. Accordingly,

$$\lim_{t \to \infty} \left\| N_{e,x,t} - N_{f,x,t} \right\| \leq C \qquad (8.4)$$

where C is a constant.

Stability often depends on the time-step size used. If the numerical solution is stable for any time step less than a certain value, then the solution is conditionally stable. If the solution is stable, regardless of the time step, then it is unconditionally stable. If the solution is unstable, regardless of the time step, then it is unconditionally unstable.

A non-stable, unconditional scheme cannot converge in general, but the analogues of individual time differences in an unstable system may converge and may be consistent. In other words, the consistence and convergence of the individual analogues does not guarantee stability. On the other hand, stability is guaranteed if the scheme is generally convergent and its analogues for the finite differences are convergent and consistent.

8.2 Exercises

Q1. Discuss numerical convergency.

Q2. Discuss numerical stability.

Q3. Discuss numerical consistency.

Q4. Define order of approximation.

Q5. Define truncation error.

Q6. Discuss the relationship between the truncation error and order of approximation.

Q7. Why do we truncate Taylor Expansion though we know that this will lessen the accuracy?