The Existence and the Uniqueness Theorem

<u>Picard iteration method</u> (successive approximation):

طريقة بيكارد (التقريبات المتعاقبة)

let $y' = f(t, y)$ differential equation of first order, with initial condition $y(t_0)$.

Picard start with $\phi_0(t) = 0$

التي تحقق الشرط الابتدائي

ولتحقيق تقريبات متعاقبة نكون $\phi_1(t)$ بالاعتماد على $\phi_0(t)$

such that $\phi_1(t) = y_0 + \int_{t_0}^{t} f(s, \phi_0(s))\,ds$

بصورة عامة : $\phi_j(t) = y_0 + \int_{t_0}^{t} f(s, \phi_j(s))\,ds$ ، $j = 1, 2, \dots$

then $\phi_n(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))\,ds$ ، $n \in \mathbb{N}$

هذه التقريبات المتعاقبة تمثل حل للمعادلة التفاضلية $y' = f(t, y)$ وهو حل تقريبي لأنها متسلسلة نقطعها عند حد معين

Example 1: Solve the initial value problem

$$y' = ty + 1 \quad , \quad y(0) = 0$$

by Picard iteration method

Solution: $\quad \phi_0(t) = 0 \quad , \quad t_0 = 0$

$$\phi_1(t) = 0 + \int_0^t s\phi_0(s) + 1 \, ds = \int_0^t s(0) + 1 \, ds = \int_0^t 1 \, ds$$

$$= t$$

$$\phi_2(t) = 0 + \int_0^t s\,\phi_1(s) + 1 \, ds = \int_0^t s(s) + 1 \, ds = \int_0^t s^2 + 1 \, ds$$

$$= \frac{s^3}{3} + s \Big]_0^t = \frac{t^3}{3} + t$$

$$\phi_3(t) = 0 + \int_0^t s\,\phi_2(s) + 1 \, ds = \int_0^t s\left(\frac{s^3}{3} + s\right) + 1 \, ds$$

$$= \frac{s^5}{3\cdot 5} + \frac{s^3}{3} + s \Big]_0^t = \frac{t^5}{3\cdot 5} + \frac{t^3}{3} + t$$

$$\therefore \phi_n(t) = \frac{t}{1\cdot 1} + \frac{t^3}{3\cdot 1} + \frac{t^5}{3\cdot 5} + \cdots + \frac{t^{2n-1}}{3\cdot 5 \cdots (2n-1)}$$

$$= \sum_{k=1}^{\infty} \frac{t^{2k-1}}{3\cdot 5 \cdots (2k-1)}$$

Example 2: Find the solution $y' = -y$, $y(0) = 1$
by 3-successive approximation

Solution: $\phi_0(t) = 1$, $t_0 = 0$, $f(t,y) = -y$

$$\phi_j(t) = y_0 + \int_{t_0}^{t} f(s, \phi_{j-1}(s))\,ds \qquad j = 1, 2, \dots$$

$$\phi_1(t) = 1 + \int_0^t -\phi_0(s)\,ds = 1 + \int_0^t -1\,ds = 1 - \int_0^t ds = 1 - t$$

$$\phi_2(t) = 1 + \int_0^t -\phi_1(s)\,ds = 1 - \int_0^t 1 - s\,ds = 1 - (s - \frac{s^2}{2})\Big|_0^t$$

$$= 1 - (t - \frac{t^2}{2})$$

$$\phi_3(t) = 1 + \int_0^t -\phi_1(s)\,ds = 1 - \int_0^t 1 - (s - \frac{s^2}{2})\,ds$$

$$= 1 - (s - (\frac{s^2}{2} - \frac{s^3}{6}))\Big|_0^t = 1 - (t - \frac{t^2}{2} + \frac{t^3}{6})$$

$$= 1 - t + \frac{t^2}{2} - \frac{t^3}{6}$$

Example 3: $y' = 2x - \frac{y}{x}$, $y(1) = 2$

Solution: $x_0 = 1$, $\phi_0 = 2$

$$\phi_j(x) = y_0 + \int_{x_0}^{x} f(s, \phi_{j-1}(s))\,ds$$

$$f(x,y) = y' = 2x - \frac{y}{x}$$

$$\therefore f(s, \phi_{j-1}(s)) = 2s - \frac{\phi_{j-1}(s)}{s}$$

$$\phi_j(x) = 2 + \int_1^x (2s - \frac{\phi_{j-1}(s)}{s})\,ds$$

if $j=1 \Rightarrow \phi_1(x) = 2 + \int_1^x (2s - \frac{\phi_0(s)}{s})\,ds$

$= 2 + \int_1^x (2s - \frac{\phi_0(s)}{s})\,ds$

$= 2(\frac{2}{2}s^2 - 2\ln s)\Big|_1^x$

$= 2 + ((x^2 - \ln x^2) - (1-0)) = 1 + x^2 - \ln x^2$

$j=2 \Rightarrow \phi_2(x) = 2 + \int_1^x (2s - \frac{1+s^2-\ln s^2}{s}\,ds)$

$\vdots$

Example: Find the ~~general~~ particular solution of the I.V.P.

$$y'' + w^2 y = f(t,y)$$

Solution: $y^2 + w^2 y = 0$

$m^2 + w^2 = 0 \Rightarrow m_1 = wi,\ m_2 = -wi$

$y = A\cos wt + B\sin wt$

$\phi_1 = \cos wt,\ \phi_2 = \sin wt$

$$W(\phi_1,\phi_2)(t) = \begin{vmatrix} \cos wt & \sin wt \\ -w\sin wt & \cos wt \end{vmatrix} = w\cos^2 wt + w\sin^2 wt = w \neq 0$$

$$u_1(t) = -\int_{t_0}^{t} \frac{\phi_2(s) f(s,y)}{a_0(s) W(\phi_1,\phi_2)}\,ds$$

$$= -\int_0^t \frac{\sin ws \cdot f(s,y)}{1\cdot w}\,ds = -\frac{1}{w}\int_0 f(s,y)\sin w.s\,ds$$

$$u_2(t) = \int_{t_0}^{t} \frac{\phi_1(s) f(s,y)}{a_0(s). W(\phi_1,\phi_2)}\,ds = \frac{1}{w}\int_0^t \cos ws . f(s,y)\,ds$$

$\therefore\ \psi_p = u_1\phi_1 + u_2\phi_2 = -\frac{\cos wt}{w}\int_0^t f(s,y)\sin ws\,ds + \frac{\sin wt}{w}\int_0^t \cos ws\, f(s,y)\,ds$

$$\psi_p = \frac{1}{w}\int_0^t f(s,y)\left[\cos ws \cdot \sin wt - \sin ws \cdot \cos wt\right] ds$$

$$= \frac{1}{w}\int_0^t f(s,y)\left[\sin(t-s)\right] ds$$