المناقشة ١٠/٢/٢١ ٢٠

Second order Linear differentiel Equation

$$y'' + a_1(t)y' + a_0(t)y = b(t)$$

where a_1, a_0, b are given functions on interval $I \subset \mathbb{R}$

1) homogenouse iff $b(t) = 0$.
2) constant coefficients if a_1, a_0 are constants.
3) variable coefficients if either a_1 or a_0 is not constants

Ex :

① second order, linear, homogenous, constant coefficients

$$y'' + 5y' + 6 = 0$$

② second order, nonhomogenous, linear, constant coefficients

$$y'' + 3y' + y = \cos(3t)$$

③ Second order, linear, nonhomogenouse, variable coefficients

$$y'' + 2ty' - \ln(t)\,y = e^{3t}$$

Ex: Find the differential equation satisfied by the family of functions

$$y(t) = c_1 e^{4t} + c_2 e^{-4t}$$

, where c_1, c_2 are arbitrary constants

Solution: compute c_1:

$$c_1 = y e^{-4t} - c_2 e^{-8t}$$

compute the derivative of y:

$$y' = 4c_1 e^{4t} - 4c_2 e^{-4t}$$

$$y' = 4(y e^{-4t} - c_2 e^{-8t}) e^{4t} - 4c_2 e^{-4t}$$

$$y' = 4y + (-4-4) c_2 e^{-4t} = 4y - 8c_2 e^{-4t}$$

$$\therefore c_2 = \frac{1}{8}(4y - y') e^{4t}$$

يعوض في c_1

$$c_1 = y e^{-4t} - \frac{1}{8}(4y - y') e^{4t} e^{-8t}$$

$$\therefore c_1 = \frac{1}{8}(4y + y') e^{-4t}$$

نشتق c_2 بالنسبة الى t

$$\Rightarrow 0 = c_2' = \frac{1}{2}(4y - y') e^{4t} + \frac{1}{8}(4y' - y'') e^{4t}$$

$$\Rightarrow 4(4y - y') + (4y' - y'') = 0$$

$$\therefore y'' - 16y = 0$$

Ex: Find the differential equation satisfied by the family of functions

$$y(t) = \frac{c_1}{t} + c_2 t \quad , c_1, c_2 \in R$$

Solution:

compute $y' \Rightarrow y' = -\frac{c_1}{t^2} + c_2 t$

$c_2 = y' + \frac{c_1}{t^2} \Rightarrow y = \frac{c_1}{t} + (y' + \frac{c_1}{t^2})t$

$y = \frac{c_1}{t} + ty' + \frac{c_1}{t} \Rightarrow y = \frac{2c_1}{t} + ty'$

$\frac{2c_1}{t} = y - ty' \Rightarrow 2c_1 = ty - t^2 y'$

$0 = (2c_1)' = (ty - t^2y')' = y + ty' - 2ty' - t^2y''$

$\therefore \; t^2 y'' + ty' - y = 0$

Ex: Find the differential equation satisfied by the family of functions

$y(x) = c_1 x + c_2 x^2$, where c_1, c_2 are arbitrary constants

$x^2 y'' - 2xy' + 2y = 0$ الجواب

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Ex: Find Picard iteration to solve $y' = 2y + 3$

$y(0) = 1$

solution: $y_{n+1}(t) = y_0 + \int_0^t f(s, y_n(s))\,ds$

$y_{n+1}(t) = 1 + \int_0^t 2y_n(s) + 3\,ds$

$y_1(t) = 1 + \int_0^t (2y_0 + 3)\,ds = 1 + \int_0^t 5\,ds = 1 + 5t$

$y_2(t) = 1 + \int_0^t 2y_1 + 3\,ds = 1 + \int_0^t 2(1+5s) + 3\,ds = 1 + 5t + 5t^2$

$y_3(t) = 1 + \int_0^t 2(1 + 5s + 5s^2) + 3\,ds = 1 + 5t + 5t^2 + \frac{10}{3}t^3$

$y_3(t) = 1 + 5t + 5t^2 + \frac{5(2)}{3}t^3$

$= 1 + 5\frac{t}{1!} + 5(2)\frac{t^2}{2!} + 5(2^2)\frac{t^3}{3!}$

$= 1 + \frac{5}{2}\frac{2t}{1!} + \frac{5}{2}\frac{(2t)^2}{2!} + \frac{5}{2}\frac{(2t)^3}{3!} = 1 + \frac{5}{2}\left(2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right)$

$\therefore y_N(t) = 1 + \frac{5}{2}\left(2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots + \frac{(2t)^N}{N!}\right) = 1 + \frac{5}{2}\sum_{k=1}^{N}\frac{(2t)^k}{k!}$

$= 1 + \frac{5}{2}(e^{2t} - 1) = \frac{5}{2}e^{2t} - \frac{3}{2}$

$\underline{Ex}$: $\dot{y} = ay + b$, $y(0) = \hat{y}_0$, $a, b \in \mathbb{R}$

$\underline{\text{Solution}}$: $y(t) = \hat{y}_0 + \int_0^t (ay(s) + b)\,ds$

$$y_1(t) = \hat{y}_0 + \int_0^t (a y_0(s) + b)\,ds = \hat{y}_0 + \int_0^t a\hat{y}_0 + b\,ds$$
$$= \hat{y}_0 + (a\hat{y}_0 + b)t$$

$$y_2(t) = \hat{y}_0 + \int_0^t (a y_1(s) + b)\,ds$$
$$= \hat{y}_0 + \int_0^t a(\hat{y}_0 + (a\hat{y}_0 + b)s) + b\,ds$$
$$= \hat{y}_0 + (a\hat{y}_0 + b)t + (a\hat{y}_0 + b)\frac{at^2}{2}$$

and $y_3(t) = \hat{y}_0 + (a\hat{y}_0 + b)t + (a\hat{y}_0 + b)\frac{at^2}{2} + (a\hat{y}_0 + b)\frac{a^2t^3}{3!}$

$$y_3(t) = \hat{y}_0 + \left(\hat{y}_0 + \frac{b}{a}\right)\left(\frac{at}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!}\right)$$
$$y_N(t) \equiv \hat{y}_0 + \left(\hat{y}_0 + \frac{b}{a}\right)\left(\frac{at}{1!} + \frac{(at)^2}{2!} + \cdots + \frac{(at)^N}{N!}\right)$$
$$= \hat{y}_0 + \left(\hat{y}_0 + \frac{b}{a}\right)\sum_{k=1}^{\infty} \frac{(at)^k}{k!}$$
$$= \hat{y}_0 + \left(\hat{y}_0 + \frac{b}{a}\right)(e^{at} - 1)$$

Ex : $y' = 5ty \quad , y(0) = 1$

$$y_{n+1}(t) = 1 + \int_0^t 5s\, y_n(s)\, ds$$

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$$\vdots$$

$$y_3(t) = 1 + \frac{5}{2}t^2 + \frac{5^2}{2^3}t^4 + \frac{5^3}{2^4 3}t^6$$

Ex : $y' = 2t^4 y \qquad y(0) = 1$

$$y_{n+1}(t) = 1 + \int_0^t 2 s^4 y_n(s)\, ds$$

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$$\vdots$$

$$y_3 = 1 + \frac{2}{5}t^5 + \frac{2^2}{5^2}\frac{1}{2}t^{10} + \frac{2^3}{5^3}\cdot\frac{1}{2}\cdot\frac{1}{3}t^{15}$$
