Definition: Set of functions $g_1, g_2, \ldots, g_m$ are $\boxed{\text{independents}}$ iff there exists constants $b_1, b_2, \ldots, b_m$ not all zero such that

$$b_1 g_1(t) + b_2 g_2(t) + \cdots + b_m g_m(t) = 0 \quad \forall t \in I$$

example: Show that $e^{r_1 t}, e^{r_2 t}$, where r_1, r_2 real constant is ~~term~~ linearly independent, $r_1 \neq r_2$ on interval I

proof: نفرض يوجد ثوابت مثل b_1، b_2 بحيث

$$b_1 e^{r_1 t} + b_2 e^{r_2 t} = 0 \quad \forall t \in I$$

ضرب طرفي المعادلة بـ $e^{-r_1 t}$ فنحصل على

$$b_1 + b_2 e^{(r_2 - r_1)t} = 0$$

بإشتقاق الطرفين

$$0 + (r_2 - r_1) b_2 e^{(r_2 - r_1)t} = 0$$

ضرب طرفي المعادلة بـ $e^{-(r_2 - r_1)t}$ فنحصل على

$$(r_2 - r_1) b_2 = 0 \implies$$

لكن $r_1 \neq r_2$

$\therefore \boxed{b_2 = 0}$

بنفس الطريقة نبرهن أن $b_1 = 0$

باختصار: اخذ ثوابت اسوي تركيب خطيه نستنتج ان الثوابت تساووا اصفار

مثال اخر في المتجهات: $(1,1)$ و $(-3,2)$ مستقلات خطيا
linearly indep.

$$a_1(1,1) + a_2(-3,2) = 0$$

ونبرهن انه $a_1 = a_2 = 0$

$$\left.\begin{aligned} a_1 - 3a_2 &= 0 \\ a_1 + 2a_2 &= 0 \end{aligned}\right\} \implies \begin{aligned} a_1 &= 0 \\ a_2 &= 0 \end{aligned}$$

$\therefore$ $(1,1)$ و $(-3,2)$ مستقلات indep.

Remark: the linear operator L such that

$$L[y] = a_0 y'' + a_1 y' + a_2 y$$ is linear, that is

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

Proof:

$$L[c_1 y_1 + c_2 y_2] = a_0 (c_1 y_1'' + c_2 y_2'') + a_1 (c_1 y_1' + c_2 y_2') + a_2 (c_1 y_1 + c_2 y_2)$$

$$= c_1 (a_0 y_1'' + a_1 y_1' + a_2 y_1) + c_2 (a_0 y_2'' + a_1 y_2' + a_2 y_2)$$

$$= c_1 L[y_1] + c_2 L[y_2]$$

Theorem: If a_0, a_1, a_2 continuous functions on interval I, $a_0(t) \neq 0$ $\forall t \in I$ then

$$L[y] = a_0 y'' + a_1 y' + a_2 y = 0$$

has two linearly independent solutions on I. Morover if ϕ solution of $L[y] = 0$ there exists c_1, c_2 s.t

$$\phi(t) = c_1 \phi_1(t) + c_2 \phi_2(t) \quad \forall \; t \in I$$

Proof: Let $t_0 \in I$, $\exists \, \phi_1$ satisfy $\phi_1(t_0) = 1$, $\phi_1'(t_0) = 0$

and ϕ_2 satisfy $\phi_2(t_0) = 0$, $\phi_2'(t_0) = 1$

To show that ϕ_1, ϕ_2 linearly independent:

$\exists \, b_1, b_2$ s.t $b_1 \phi_1(t_0) + b_2 \phi_2(t_0) = 0 \quad \forall t_0 \in I$

$$b_1 \cdot 1 + b_2 \cdot 0 = 0 \Rightarrow b_1 = 0$$

Since ϕ_1, ϕ_2 solutions then قابلة للاشتقاق يعني

$b_1 \phi_1' + b_2 \phi_2' = 0$

$b_1 \cdot 0 + b_2 \cdot 1 = 0 \Rightarrow b_2 = 0$

$\boxed{b_1 = b_2 = 0} \Rightarrow \phi_1, \phi_2$ linearly indep.

لنكن ϕ اي حل:

$\phi(t) = c_1 \phi_1 + c_2 \phi_2$

$\phi(t_0) = c_1 \phi_1(t_0) + c_2 \phi_2(t_0)$

$\alpha = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = \alpha$

and $\phi'(t_0) = c_1 \phi_1'(t_0) + c_2 \phi_2'(t_0)$

$B = c_1 \cdot 0 + c_2 \cdot 1 \Rightarrow c_2 = B$

Define the function

$\psi(t) = \alpha \phi_1(t) + B \phi_2(t) \quad \forall t \in I$

$\psi(t)$ Solution of $L[y] = 0$ on I

$\psi(t_0) = \alpha \phi_1(t_0) + B \phi_2(t_0)$

$= \alpha \cdot 1 + B \cdot 0 \Rightarrow \psi(t_0) = \alpha = \phi(t_0)$

$\psi'(t_0) = \alpha \phi_1'(t_0) + B \phi_2'(t_0)$

$= \alpha \cdot 0 + B \cdot 1 \Rightarrow \psi'(t_0) = B = \phi'(t_0)$

وعليه يكون كل من ψ و ϕ تمثل حلاً للمعادلة $L[y]=0$ وحسب مبرهنة الوجود والوحدانية نحصل على ان

$\phi(t) = \psi(t) = c_1 \phi_1(t) + c_2 \phi_2(t)$

Theorem: If a_0, a_1, a_2 continuous function on I

$a_0(t) \neq 0 \ \forall t \in I$, if ϕ_1, ϕ_2 linearly independent fun.

on $L[y] = a_0(t)y'' + a_1(t)y' + a_2(t)y$ on I

then $\forall \phi$ $L[y] = 0$

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) \quad \forall t \in I$$

يعني الثوابت الوحيده c_1, c_2 (يقال) ل ϕ_1, ϕ_2 بأنها

تمثل مجموعة الحلول الاساسية).

Proof: let ϕ solution of $L[y] = 0$ on I

$t_0 \in I$ s.t. $\phi(t_0) = \alpha$, $\phi'(t_0) = \beta$

by ϕ_1, ϕ_2 linearly indep. $\Rightarrow W(\phi_1, \phi_2) \neq 0$

$\therefore W(\phi_1, \phi_2)(t_0) \neq 0$

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) = \alpha$$

$$c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = \beta$$

حل بطريقة كرامر

$$C_1 = \frac{\begin{vmatrix} \alpha & \phi_2 \\ \beta & \phi_2' \end{vmatrix}}{W} \Rightarrow c_1 = \frac{\alpha\phi_2' - \beta\phi_2}{W}$$

$$C_2 = \frac{\begin{vmatrix} \phi_1 & \alpha \\ \phi_1' & \beta \end{vmatrix}}{W} \Rightarrow C_2 = \frac{\beta\phi_1 - \alpha\phi_1'}{W}$$

حسب مبرهنة الوحدانية:

نحصل على:

$$\psi(t) = \phi(t)$$

$$\psi(t_0) = \phi(t_0) = \alpha$$

$$\psi'(t_0) = \phi'(t_0) = \beta$$

$$\phi = \psi$$