Lecture (9)

Time – Stepping Schemes and CFL Criterion

9.1 The differences between the explicit and the implicit solution of finite-difference method

The method of explicit solution expresses an unknown value at a certain grid point in terms of known values at neighboring points. It is a computationally simple method for solving, in which the values of the dependent variable (the solution) can be obtained directly. This method has a drawback, that the time step should be necessarily very small because the process is acceptable only for $k \leq \frac{1}{2}h^2$ and h must be small in order to obtain reasonable accuracy, this leads to the arithmetic operations being very large.

The implicit solution method expresses the solution in the form of simultaneous equations with the same number of grid points at the first row in terms of the known initial and boundary conditions values. This method requires solving the simultaneous equations to obtain the numerical solution. This method reduces the overall volume of calculations and is acceptable (i.e., it is convergent and stable) for all $r=k/h^2$ values. In this method, k does not have to be very small in comparison to h, and they can be equal.

9.2 Advection-Diffusion Equations

They are common type of partial differential equations, which are used in atmospheric sciences and numerical weather prediction systems. The qualitative continuity equation can be written in the form of the advection-diffusion equation as follows:

$$\frac{\partial N}{\partial t} + \frac{\partial (uN)}{\partial x} - \frac{\partial}{\partial x}\left(K_{h,xx}\frac{\partial N}{\partial x}\right) = 0 \qquad (9.1) \qquad west - east\ eq.$$

$$\frac{\partial N}{\partial t} + \frac{\partial (vN)}{\partial y} - \frac{\partial}{\partial y}\left(K_{h,yy}\frac{\partial N}{\partial y}\right) = 0 \qquad (9.2) \qquad south - north\ eq.$$

$$\frac{\partial N}{\partial t} + \frac{\partial (wN)}{\partial z} - \frac{\partial}{\partial z}\left(K_{h,zz}\frac{\partial N}{\partial z}\right) = 0 \qquad (9.3) \qquad vertical\ eq.$$

where N is the gas concentration,

$K_{h,xx}, K_{h,yy}$ *and* $K_{h,zz}$ are eddy diffusion coefficients in x-, y-, and z- directions, respectively. The subscript h indicates that the diffusion coefficient for energy (eddy thermal diffusivity) is used.

9.3 Courant-Friedrichs-Lewy Stability Criterion (CFL criterion)

While the explicit schemes are conditionally stable (stable at any time step below a certain value), some implicit schemes may be stable unconditionally (stable regardless of the time step). The limits of time step for an explicit solution of the advection equation (in west-east direction) can be obtained by using CFL criterion, by neglecting the diffusion term from equation (9.1). When the resulting equation is explicitly solved, stability is generally guaranteed when the following CFL standard is achieved:

$$k < \frac{\Delta x_{min}}{|u_{max}|} \qquad (9.4)$$

where $|u_{max}|$ is the maximum west-east wind speed, and Δx_{min} is the minimum west east grid – cell length in the domain.

Homework: If the maximum wind speed is 20 m/s and the minimum grid-cell length is 5 km. Find the maximum time step that the CFL criterion predicts for maintaining stability.

Now, if the advection term is neglected from equation 9.1, the equation will simplified to the diffusion equation and its CFL criterion will be:

$$k < \frac{\Delta {x^2}_{min}}{|K_{max}|} \qquad (9.5)$$

where K_{max} is the largest eddy diffusion coefficient in the domain.

For a typical vertical eddy diffusion coefficient of value (50 $m^2\ s^{-1}$), the stability criterion suggests that the time step for eddy diffusion within a height of 100 m should be less than 200s. For a typical horizontal eddy diffusion coefficient (2500 $m^2\ s^{-1}$) the time step for eddy diffusion through a 5 km cell should be less than 10000s.

Homework

1. Discuss this statement: h must be small in order to obtain reasonable accuracy, this leads to the arithmetic operations being very large.
2. Discuss this statement: The implicit method reduces the overall volume of calculations.