Lecture 5

The Gravitational Force

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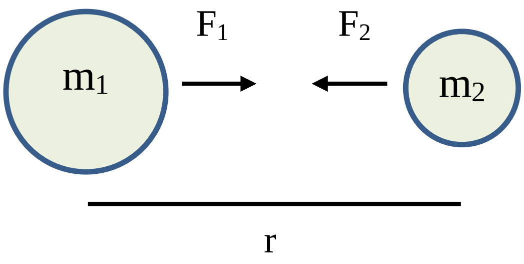
Newton's law of the universal gravitation states, "Any two elements of mass in the universe attract each other with a force proportional to their masses and inversely to the square of the distance between them."



Newton's law can be written in a vectorial form as:

$$\vec{g} = -G\frac{m_1 m_2}{r^3}\,\vec{r}$$

where $\vec{g}$ is the attraction of m_1 on m_2 (force of gravitation)

$\vec{r}$ is the position vector from m_1 to m_2

G is the universal gravitational constant = 6.66×10^{-11} Nm^2kg^{-2}

If we assume m_2=1 kg ,

$$\vec{g} = -G\frac{m_1}{r^3}\,\vec{r}$$

If m_1 = M "total mass of Earth is equal to 5.988×10^{24} kg"

The acceleration due to the gravitational force at the surface of Earth (r=a=6378 km):

$$\vec{g}_* = -G\frac{M}{a^2}\,\vec{r}$$

At some altitude Z above the surface of the earth, the acceleration due to the gravitational force is:

$$\vec{g}_* = -G\frac{M}{(a+z)^2}\,\vec{r}$$

$\vec{r}$: is the position vector from the center of Earth to the parcel in the atmosphere.

$\vec{g}_*$ is directed toward the center of Earth.

Balanced Forces

Consider a ball of mass m is attached to a string and whirled through a circle of radius r at a constant angular velocity ω. From the point of view of an observer in inertial space, the speed of the ball is constant, but its direction of travel is continuously changing so that its velocity is not constant. To compute the acceleration we consider the change in velocity $\delta\vec{V}$ that occurs for a time increment δt during which the ball rotates through an angle $\delta\theta$ as shown in Fig. 5.1. Because $\delta\theta$ is also the angle between the vectors $\vec{V}$ and $\vec{V} + \delta\vec{V}$, the magnitude of $\delta\vec{V}$ is just $|\delta\vec{V}| = |\vec{V}|\,\delta\theta$ (the length of arc formula). If we divide by δt and note that in the limit $\delta t \to 0$, δV is directed toward the axis of rotation, we obtain:

$$\frac{\delta\vec{V}}{\delta t} = |\vec{V}|\frac{\delta\theta}{\delta t}\left(-\frac{\vec{r}}{r}\right) \qquad (5.1)$$

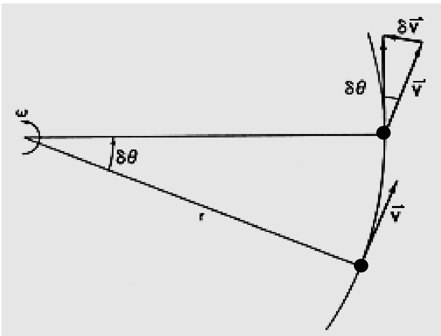


Fig. 5.1 Centripetal acceleration is given by the rate of change of

the direction of the velocity vector, which is directed toward the axis of rotation, as illustrated here by $\delta\vec{V}$

However, $|\vec{V}| = \omega r$ and $\delta\theta/\delta t = \omega$, so that:

$$\frac{\delta\vec{V}}{\delta t} = -\omega^2 r \qquad (5.2)$$

Therefore, viewed from fixed coordinates the motion is one of uniform acceleration directed toward the axis of rotation and equal to the square of the angular velocity times the distance from the axis of rotation. This acceleration is called centripetal acceleration. It is caused by the force of the string pulling the ball.

Now suppose that we observe the motion in a coordinate system rotating with the ball. In this rotating system, the ball is stationary, but there is still a force acting on the ball, namely the pull of the string. Therefore, in order to apply Newton's second law to describe the motion relative to this rotating coordinate system, we must include an additional apparent force, the centrifugal force, which just balances the force of the string on the ball. Thus, the centrifugal force is equivalent to the inertial reaction of the ball on the string and just equal and opposite to the centripetal acceleration.

To summarize, observed from a fixed system the rotating ball undergoes a uniform centripetal acceleration in response to the force exerted by the string. Observed from a system rotating along with it, the ball is stationary and the force exerted by the string is balanced by a centrifugal force.

Gravity Force

An object at rest on the surface of the earth is not at rest or in uniform motion relative to an inertial reference frame except at the poles. Rather, an object of unit mass at rest on the surface of the earth is subject to a centripetal acceleration directed toward the axis of rotation of the earth given by $-\Omega^2\vec{R}$, where $\vec{R}$ is the position vector from the axis of rotation to the object and $\Omega = 7.292 \times 10^{-5}\ rad\ s^{-1}$ is the angular speed of rotation of the earth.

Viewed from a frame of reference rotating with the earth, however, a geopotential surface is everywhere normal to the sum of the true force of gravity, g^*, and the centrifugal force $\Omega^2\vec{R}$ (which is just the reaction force of the centripetal acceleration). A geopotential surface is thus experienced as a level surface by an object at rest on the rotating earth. Except at the poles, the weight of an object of mass m at rest on such a surface, which is just the reaction force of the earth on the object, will be slightly less than the gravitational force mg^* because, as illustrated in Fig. 5.2, the centrifugal force partly balances the gravitational force. It is, therefore, convenient to combine the effects of the gravitational force and centrifugal force by defining gravity g such that:

$$\vec{g} \equiv -g\,k \equiv \vec{g}^* + \Omega^2\vec{R}$$

where k designates a unit vector parallel to the local vertical. Gravity, g, sometimes referred to as "apparent gravity," will here be taken as a constant ($g = 9.81\ ms^{-2}$). Except at the poles and the equator, g is not directed toward the center of the earth, but is perpendicular to a geopotential surface as indicated by Fig. 5.2.

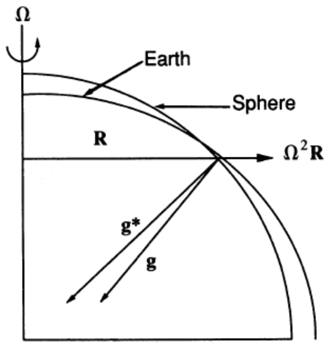
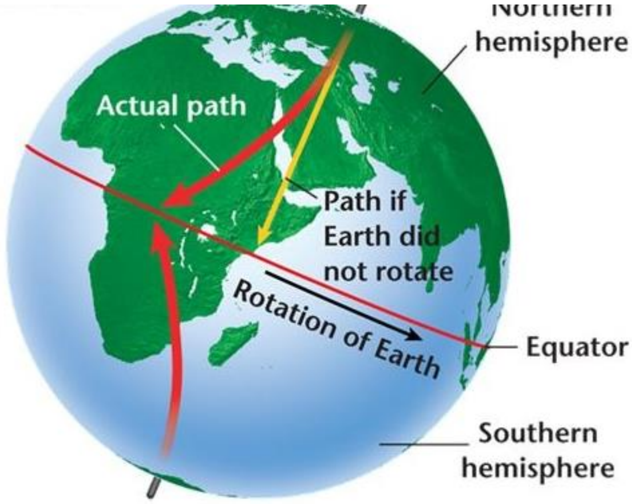


Fig. 5.2

The Coriolis Force

The Coriolis Force

- ✓ The Coriolis Effect is the apparent deflection of a moving object when viewed from a rotating frame of reference. The effect is named after the French scientist who first described it in 1835. The effect is most commonly associated with the rotation of the Earth. Objects that travel in a straight path appear to veer right in the northern hemisphere and left in the southern hemisphere. In actuality the object stays on a straight path and only appears to curve due to our rotating due to our rotating frame of reference on the Earth.



- ✓ The Coriolis force is another **apparent** force, which occur because of the rotation of Earth when viewed from the coordinates that rotate with the earth itself, not from the inertial coordinates.
- ✓ If a person 1 rolls a ball toward person 2 on a frictionless surface at a uniform speed in a straight line. It will appear (for person 1) to deflect to the right (see figure 8.1) {in a direction opposite the rotation}.

  To a person on the ground, the ball appears to travel in straight line (non-accelerating), but from the coordinate system of merry-go-round, the ball accelerates (velocity changes).

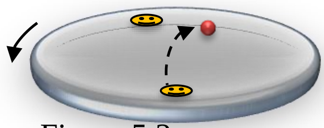
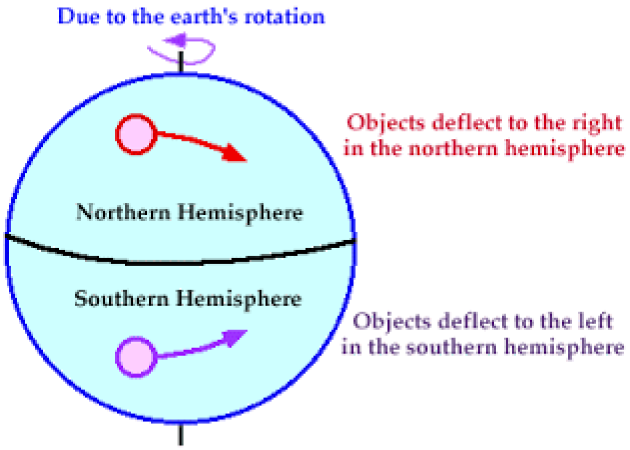


Figure 5.3 merry-go-round

- ✓ The Coriolis force acts perpendicular to the direction of the velocity vector and in direct proportion to its magnitude.
- ✓ The victorial formula of Coriolis force is:

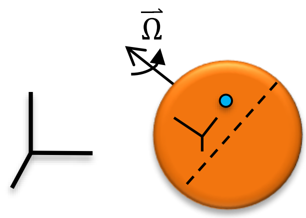
$$\vec{F}_{cor} = -2\vec{\Omega} \times \vec{V} \qquad (5.3)$$

- ✓ In N.H., $\vec{\Omega}$ is counterclockwise, the Coriolis force acts to the right of the velocity vector.
- ✓ The Coriolis force changes only the direction of the particle's motion, not its speed, because the force is always perpendicular to the direction of the motion (just like the centripetal force in uniform circular motion).



Components of Coriolis Force

- ✓ Consider a parcel of air (unit mass) initially is at rest in a rotating frame. At this moment the centrifugal force is acting along with other forces.
- ✓ Suppose now that the parcel is suddenly <u>set in motion</u> toward the east. The parcel is now rotating faster than the earth, and thus the centrifugal force will be <u>stronger</u>.



- ✓ The total centrifugal force is then composed of that from earth's rotation plus that due to the eastward motion, i.e.:

$$(\Omega + \frac{u}{R})^2 \vec{R} = \Omega^2 \vec{R} + \frac{2\Omega u}{R} \vec{R} + \frac{u^2}{R^2} \vec{R} \qquad (5.4)$$

where u is the eastward speed, $\vec{R}$ is the position vector from the axis of rotation.

- For synoptic scale motion, $/u/ \ll \Omega R \quad or \quad \frac{/u/}{R} \ll \Omega$
- The last term can be neglected:

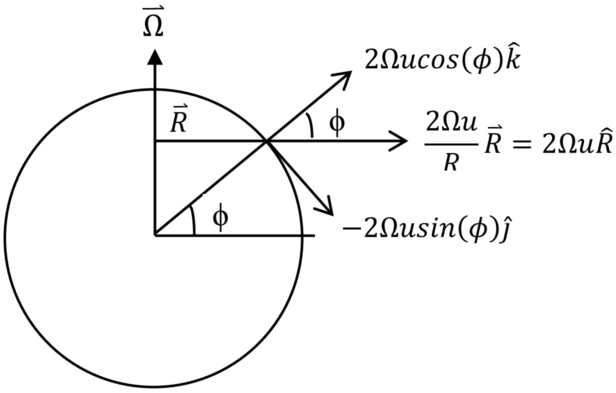
$$\frac{2\Omega u}{R}\vec{R} + \frac{u^2}{R^2}\vec{R} = \frac{2\Omega u}{R}\vec{R}\left(1 + \frac{u}{2\Omega R}\right) = \frac{2\Omega u}{R}\vec{R}$$

The result is the Coriolis force due to the motion along a latitude circle.

- Let's look at the component of this vector, we see that due to the Coriolis force alone:

$$\left(\frac{dv}{dt}\right)_{cor} = -2\Omega u\, sin(\phi)$$

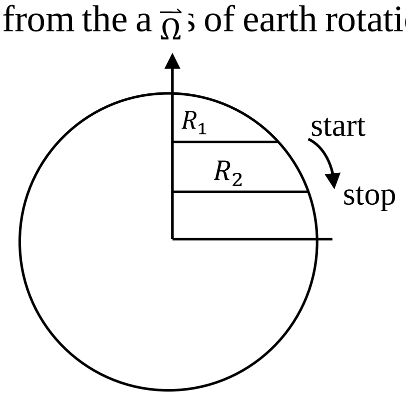
$$\left(\frac{dw}{dt}\right)_{cor} = 2\Omega u\, cos(\phi)$$



That means if a parcel moves to the east in the horizontal plane in N.H., it deflects southerly and upward; the vice versa for a parcel moves from the east to the west.
If you use numbers, you will find that the vertical component of Coriolis force is much less than the gravity, hence it is not causes large change in the vertical location of the parcel, but the horizontal components could be large in comparison to the horizontal forces (horizontal pressure gradient force is another dominant force).

✓ Now let's repeat this exercise for a parcel displaced toward the equator. What's different here?

- R is the distance from the parcel from the axis of earth rotation changes (increases).



Because the forces acting on the parcel are central, the torque is zero-there is no torque in the east-west direction. Therefore, the parcel conserves angular momentum. So if R increases what will happen to the velocity?
Initially, it has no motion tangential to the direction of rotation. The angular momentum is ΩR^2, so as R increases, the Ω of the parcel has to decrease → the absolute eastward velocity decreases, the parcel starts to deflect toward west. So, the particle moving south is deflected to the west.

Air tends to flow from low pressure areas to high pressure areas. However, when we observe the actual flow of air masses notice a tendency to flow perpendicular to low pressure areas. This phenomenon can be explained by the Coriolis Effect. As the following video shows, we would predict the flow of air to go directly towards the low pressure zone. However, since the Earth is rotating, we observe an air flow perpendicular to the expected path. Meteorologists need to account for this effect when making weather forecasts.