Geometric Distribution

The geometric distribution represents the number of failures before you get a success in a series of Bernoulli trials. In other words, the Geometric pmf tells us the probability that the first occurrence of success requires x number of independent trials, each with success probability p. If the probability of success on each trial is *p*, then the probability that the x^{th} trial (out of x trials) is the first success is:

This discrete probability distribution is represented by the probability density function:

$$P(X = x) = \begin{cases} P(1-P)^{x-1} & , \quad x = 1, 2, \dots \\ 0 & , \quad o.w \end{cases}$$

For example,

1. Toss a coin repeatedly. Let X = number of tosses to first head.

2. It is known that 20% of products on a production line are defective. Products are inspected until first defective is encountered. Let X = number of inspections to obtain first defective 4.

Geometric Distribution Conditions:

1. An experiment consists of repeating trials until first success.

2. Each trial has two possible outcomes;

(a) A success with probability p

(b) A failure with probability q = 1 − p.

3. Repeated trials are independent. X = number of trials to first success.

The properties of the Geometric distribution

$1.E(X) = \frac{1}{p}$

$2.\ ver(X) = \frac{1-p}{p^2}$

Proof:

$$E\ (X) = \sum_{\forall X} x\, p(x)$$

$$= \sum_{\forall X} x\, pq^{x-1}$$

$$= p \sum_{\forall X} x\, q^{x-1}$$

$$= p \sum_{\forall X} \frac{d}{dq} q^x$$

$$= p \frac{d}{dq} \sum_{x=1}^{\infty} q^x = p \frac{d}{dq} (q + q^2 + q^3 + \cdots)$$

$$= p \frac{d}{dq} \left(\frac{q}{1-q} \right)$$

$$= p\ \frac{(1-q) - q\ (-1)}{(1-q)^2}$$

$$= p \frac{1-q+q}{(1-q)^2} = p \frac{1}{(1-q)^2} = \frac{1}{p}$$

2. var (x)

$$var(x) = E(X)^2 - [E(X)]^2$$

$$E(X)^2 = E\big(X(X-1)\big) + E(X)$$

$$E\big(X(X-1)\big) = \textstyle\sum_{\forall X}\big(X(X-1)\big)\, p(x)$$

$$= \sum_{\forall X}\big(X(X-1)\big)\, pq^{x-1}$$

$$= pq \sum_{x=2}^{\infty}\big(X(X-1)\big)\, q^{x-2}$$

$$= pq \sum_{x=2}^{\infty} \frac{d^2}{dq^2}\, q^{x-2}$$

$$= pq \frac{d^2}{dq^2} \sum_{x=2}^{\infty} q^x$$

$$= pq \frac{d^2}{dq^2} \sum_{x=2}^{\infty} q^x$$

$$= pq \frac{d^2}{dq^2} (q^2 + q^3 + \cdots)$$

$$= pq^3 \frac{d^2}{dq^2} (1 + q + q^2 + q^3 + \cdots)$$

$$= pq^3 \frac{d^2}{dq^2}\left(\frac{1}{1-q}\right)$$

$$= pq^3 \frac{d}{dq}\left[\frac{1}{(1-q)^2}\right]$$

$$= pq^3 \left[\frac{-2(1-q)(-1)}{(1-q)^4}\right]$$

$$= pq^3 \left[\frac{2}{(1-q)^3}\right]$$

$$= \frac{2q^3}{p^2}$$

$$E(X)^2 = \frac{2q^3}{p^2} + \frac{1}{p}$$

$$var(x) = \frac{2q^3}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q^3 + p - 1}{p^2}$$

Example. 1

Products produced by a machine has a 3% defective rate. What is the probability that the first defective occurs in the fifth item inspected?

Solution:

$$P(X = x) = \begin{cases} P(1-P)^{x-1} & , \quad x = 1, 2, \dots \\ 0 \quad , & \quad o.w \end{cases}$$

P = 0.03

P(X = 5) = P(1st 4 non-defective) P(5th defective)

$$= (\mathbf{1} - \mathbf{0.03})^{\mathbf{5-1}}(0.03)$$

$$= (\mathbf{0.97})^{\mathbf{4}}\ (0.03)$$

Example. 2:

The probability of hitting a target is (0.4) what is the probability of hitting this target on the fourth attempt.

Solution:

The probability of hitting the target on the fourth attempt means that failure in the previous three attempts, and therefore the probability is:

$$P(X = x) = \begin{cases} P(1 - P)^{x-1} & , \quad x = 1, 2, \ldots \\ 0 & , \quad o.w \end{cases}$$

P = 0.4

$$= (\mathbf{1} - \mathbf{0.4})^{\mathbf{4-1}}(0.4)$$

$$= (0.6)^{3}\ (0.4) = 0.0864$$