1.7. Logical Implication

Definition 1.7.1. (Logical implication)
We say the logical proposition r implies the logical proposition s (or s logically deduced from r) and write $r \Longrightarrow s$ if $r \rightarrow s$ is a tautology.
Example 1.7.2. Show that $(p \rightarrow t) \wedge (t \rightarrow q) \Longrightarrow p \rightarrow q$.

Solution. Let P: the proposition $(p \rightarrow t) \wedge (t \rightarrow q)$

Q: the proposition $p \rightarrow q$

p	t	q	$p \rightarrow t$	$t \rightarrow$ **q**	P	Q	$P \rightarrow Q$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Remark 1.7.3.

(i) We use $r \Longrightarrow s$ to imply that the statement $r \rightarrow s$ is true, while the statement $r \rightarrow s$ alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for "implies".

(ii) If $r \Longrightarrow s$ and $s \Longrightarrow r$, then we denote that by $r \Longleftrightarrow s$.

Example 1.7.4. Show that
(i) $r \Longrightarrow s$ if and only if $\sim r \vee s$ is tautology.
(ii) $r \Longleftrightarrow s$ if and only if $r \equiv s$.
Solution.
(i) $r \Longrightarrow s$ if and only if $r \rightarrow s$ is a tautology (by def.)
But $\sim r \vee s \equiv r \rightarrow s$ is a tautology.
Then, $r \Longrightarrow s$ if and only if $\sim r \vee s$ is tautology.

(ii) $r \Longrightarrow s$ if and only if $r \rightarrow s$ is tautology (by def.)
$s \Longrightarrow r$ if and only if $s \rightarrow r$ is tautology (by def.)
Then, $r \rightarrow s \wedge s \rightarrow r$ is tautology. Therefore, $r \equiv s$.