Let $f(x)$ be a logical proposition which depend only on x. A sentence $\forall\, x\, f(x)$ is true if and only if $f(x)$ is true no matter what value (from the universe of discourse) is substituted for x.

Example 1.8.3.

$\forall\, x : (x^2 \geq 0)$, i.e., "the square of any number is not negative."
$\forall\, x$ and $\forall\, y, (x + y = y + x)$, i.e., the commutative law of addition.
$\forall\, x, \forall\, y$ and $\forall\, z, ((x + y) + z = x + (y + z))$, i.e. the associative law of addition.

Remark .1.8.4. The *"***all***"* form, the universal quantifier, is frequently encountered in the following context: $\forall\, x\, (f(x) \Rightarrow Q(x))$,

which may be read, "For all x satisfying $f(x)$ also satisfy $Q(x)$. " Parentheses are crucial here; be sure you understand the difference between the "all" form and $\forall\, x\, f(x) \Rightarrow \forall\, x\, Q(x)$ and $(\forall\, x\, f(x)) \Rightarrow Q(x)$.

Definition 1.8.5. (The Existential Quantifier)

A sentence $\exists\, x\, f(x)$ is true if and only if there is at least one value of x (from the universe discourse of) that makes $f(x)$ is true.

Example 1.8.6.

$\exists\, x: (x \geq x^2)$ is true since $x = 0$ is a solution. There are many others.

$\exists\, x\, \exists\, y: (x^2 + y^2 = 2xy)$ is true since $x = y = 1$ is one of many solutions

Negation Rules 1.8.7. When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negative the statement.
Definition 1.8.8.
(i) $\forall x\, f(x) = \sim \exists\, x \sim f(x)$.
(ii) $\exists\, x\, f(x) = \sim \forall\, x \sim f(x)$.

Example 1.8.9. Express each of the following sentences in symbolic form and then give its negation.

(i) r: The square of every real number is non-negative.
Solution. Symbolically, r can be expressed as $\forall x \in \mathbb{R}, x^2 \geq 0$.
$\sim$r: $\sim (\forall x \in \mathbb{R}, x^2 \geq 0) \equiv \exists x \in \mathbb{R}, \sim (x^2 \geq 0) \equiv \exists x \in \mathbb{R}, x^2 < 0$.