# Lecture note on wind turbine

Prepared by Minoru Taya, Jan. 26, 2016

Main reference: "Wind energy explained:theory, design and application", by F. Manwell, J.G. McGowan and A.L. Rogers, Joh Wiley and Sons, LTD, 2002

Sub reference, "Electronic Composites", by M. Taya, Cambridge University Press, 2005.

**1. Fundamentals of wind energy harvesting**

*Linear momentum model*

$$\underline{P} = m\underline{v}$$

$m$: mass
$v$: velocity

For steady-state, conservation of mass

$$(\rho A u)_1 = (\rho A u)_4 = \dot{m} \tag{1}$$

Then,

$$T = U_1(\rho A u)_1 - U_4(\rho A u)_4 = \dot{m}(U_1 - U_4) \tag{2}$$

where, $T$: thrust

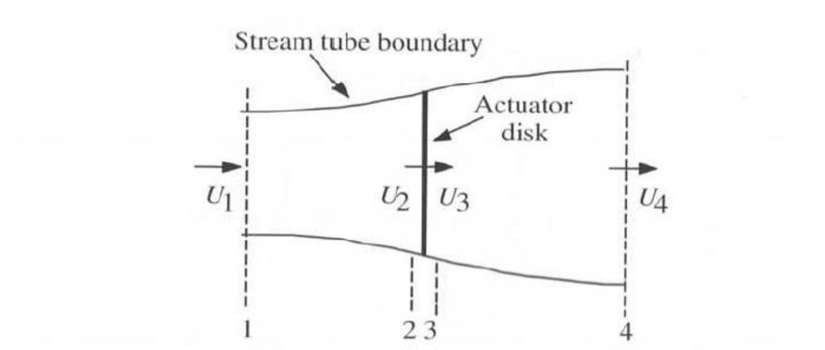


Fig.1: Actuator disk model of a wind turbine; U-mean air velocity, 1,2,3 and 4 indicate locations

Applying Bernoulli theory at control section 1, 2 (both upstream of turbine)

$$P_1 + \frac{1}{2}\rho {U_1}^2 = P_2 + \frac{1}{2}\rho {U_2}^2 \tag{3}$$

For downstream,

$$P_3 + \tfrac{1}{2}\rho {U_3}^2 = P_4 + \tfrac{1}{2}\rho {U_4}^2 \tag{4}$$

Assuming $P_1 = P_4$, $U_2 = U_3$

Thrust $T$ can be expressed as

$$T = A_2(P_2 - P_3) \tag{5}$$

Using the assumption of $P_1 = P_4$, $U_2 = U_3$

$$P_2 = P_1 + \frac{1}{2}\rho\, {U_1}^2 - \frac{1}{2}\rho {U_2}^2$$

$$P_3 = P_4 + \frac{1}{2}\rho\, {U_4}^2 - \frac{1}{2}\rho {U_3}^2$$

$$P_2 - P_3 = \tfrac{1}{2}\rho({U_1}^2 - {U_4}^2) \tag{6}$$

Substituting (6) into (5)

$$T = \tfrac{1}{2}\rho A_2({U_1}^2 - {U_4}^2) \tag{7}$$

Equating (7) to (2)

$$T = \dot{m}(U_1 - U_4) = \tfrac{1}{2}\rho A_2({U_1}^2 - {U_4}^2)$$

$$\rho A_2 U_2(U_1 - U_4) = \frac{1}{2}\rho A_2(U_1 - U_4)(U_1 + U_4)$$

$$\therefore U_2 = \tfrac{1}{2}(U_1 + U_4) \tag{8}$$

Define $a$ as the fractional decrease in wind velocity between free stream and rotor plane, then

$$a \equiv \tfrac{(U_1 - U_2)}{U_1} \tag{9}$$

$$U_2 = U_1(1 - a) \tag{10}$$

From (8) and (10)

$$\frac{1}{2}(U_1 + U_4) = U_1(1 - a)$$

$$U_4 = U_1(1 - 2a) \tag{11}$$

$aU_1$: induction velocity

Power of wind turbine, $P$ is equal to thrust ($T$) times velocity ($U_2$), from (7)

$$P = \frac{1}{2}\rho A_2\left({U_1}^2 - {U_4}^2\right)U_2 = \frac{1}{2}\rho A_2(U_1 - U_4)(U_1 + U_4)U_2 \tag{12}$$

Substituting (10), (11) into (12)

$$P = \frac{1}{2}\rho A U^3 4a(1-a)^2 \tag{13}$$

where $A_2 = A$, $U_1 = U$

Power coefficient, $C_p$ is given by

$$C_p = \frac{P}{\frac{1}{2}\rho U^3 A} = \frac{Rotor\ power}{Power\ in\ the\ wind} \tag{14}$$

From (13) and (14)

$$C_p = 4a(1-a)^2 \tag{15}$$

The maximum of $C_p$ is obtained by taking its derivative $\frac{dC_p}{da}$ and setting it to be zero.

$$\frac{dC_p}{da} = 4\{(1-a)^2 + 4a2(1-a)(-1)\} = 0$$

$$a_{max} = \frac{1}{3} \tag{16}$$

Substituting $a = \frac{1}{3}$ in (15)

$$C_{p,max} = \frac{16}{27} = 0.5926 \tag{17}$$

Substituting (10), (11) into (7)

$$\begin{aligned} T &= \frac{1}{2}\rho A_2\left({U_1}^2 - {U_4}^2\right) = \frac{1}{2}\rho A_2\left\{{U_1}^2 - {U_1}^2(1-2a)^2\right\} \\ &= \frac{1}{2}\rho A_2 {U_1}^2\{1 - (1-2a)^2\} = \frac{1}{2}\rho A {U_1}^2[4a(1-a)] \end{aligned} \tag{18}$$

where $A_2 = A$

Thrust coefficient, $C_T$ is given by

$$C_T = \frac{T}{\frac{1}{2}\rho U^2 A} = \frac{Thrust\ force}{Dynamic\ force} \tag{19}$$

$$= 4a(1-a)$$

The maximum of $C_T$ is obtained from

$$\frac{dC_T}{da} = 0 \rightarrow\ a = \frac{1}{2} \tag{20}$$

$C_T$ becomes the maximum, $C_T = 1$ at $a = \frac{1}{2}$

At $a = \frac{1}{2}$,

$$U_4 = U_1(1 - 2a) = 0$$

At maximum power output ($a = \frac{1}{3}$)

$$C_T = \frac{8}{9} \tag{21}$$

It is noted from Fig. 2, the above model (Betz) is valid only for $a \leq \frac{1}{2}$

At $a = \frac{1}{2}$, $C_{p,max} = \frac{16}{27} = 0.5926$, is theoretically the maximum rotor power efficiency.

Practical turbine efficiency, $\eta_{overall}$ is given by

$$\eta_{overall} = \frac{P_{out}}{\frac{1}{2}\rho A U^3} = \eta_{mech} C_p$$

$$\therefore P_{out} = \frac{1}{2}\rho A U^3(\eta_{mech} C_p) \tag{22}$$

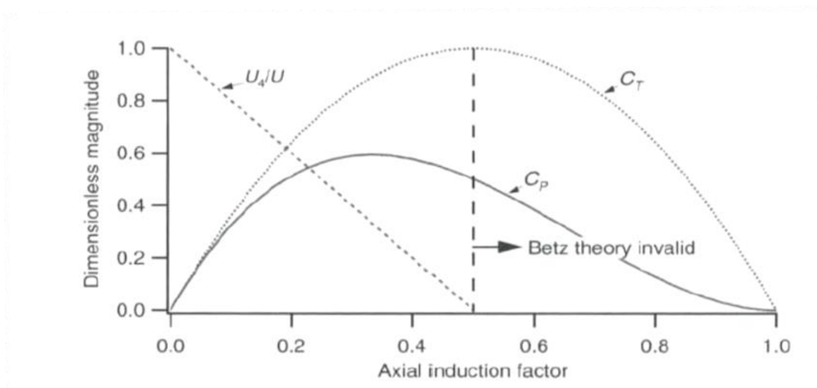


Fig. 2: Operating parameters for a Betz turbine: $U$-velocity of undisturbed air, $U_4$-air velocity behind th e rotor, $C_p$-pwer coefficient, and $C_T$-thrust coefficient

The previous model did not account for the rotating turbine blades which induces angular momentum in the wake region, see Fig. 3.

*Conservation of angular momentum*

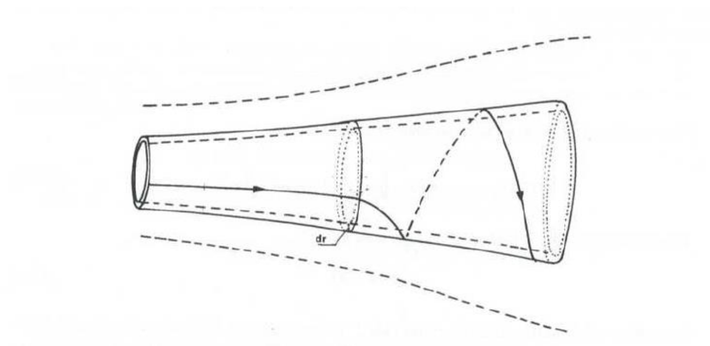


Fig. 3: Stream tube model of flow behind rotating wind turbine blade.

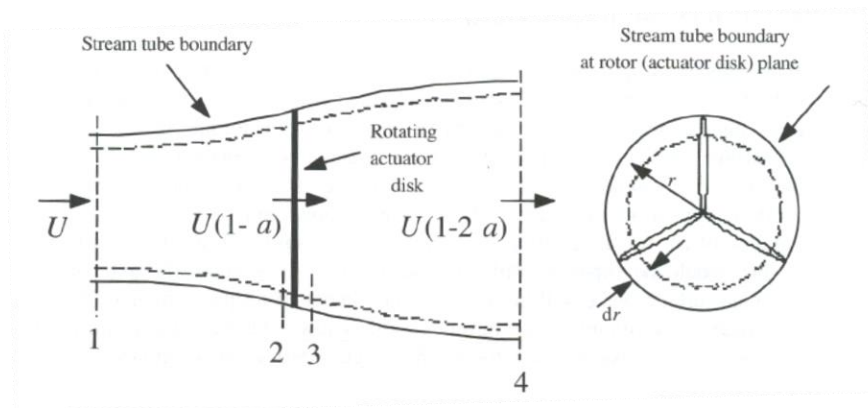


Fig. 4: Geometry for rotor analysis; $U$ – velocity of undisturbed air, $a$ - induction factor, $r$ – radius

$\omega$: angular velocity of the air flow in the wake

$\Omega$: angular velocity of turbine blade

Assume $\Omega \gg \omega$

Then, the pressure gap before and after the rotor location is estimated as

$$P_2 - P_3 = \rho\left(\Omega + \frac{1}{2}\omega\right)\omega r^2 \tag{23}$$

For the annual element ($dr$), thrust, $dT$ is given by

$$dT = (P_2 - P_3)dA = \rho\left(\Omega + \frac{1}{2}\omega\right)\omega r^2 \cdot 2\pi r\, dr \tag{24}$$

Define angular induction factor, $a'$ as

$$a' = \frac{\omega}{2\Omega} \tag{25}$$

Induced velocity at the rotor is sum of axial component $Ua$, and $r\Omega a'$.

By using (25), (24) is reduced to

$$dT = 4a'(1 + a')\frac{1}{2}\rho\Omega^2 r^2 2\pi r dr \quad (26)$$

From (18) applied to the annual area element ($2\pi r dr$)

$$dT = 4a(1 - a)\frac{1}{2}\rho U^2 2\pi r dr \quad (27)$$

Equating (26) to (27), we obtain

$$\frac{a(1-a)}{a'(1+a)} = \frac{\Omega^2 r^2}{U^2} \equiv \lambda_r^2 \quad (28)$$

where $\lambda r$ is local speed ratio. Tip speed ratio, $\lambda$ is given by

$$\lambda = \frac{\Omega R}{U} \quad (29a)$$

Intermediate speed ratio (at $r$), $\lambda_r$ is given by

$$\lambda_r = \frac{\Omega_r}{U} = \frac{\lambda r}{R} \quad (29b)$$

Let us consider the conservation of angular momentum focusing on the annual element. The torque ($Q$) expected  on the rotor is equal to the change in the angular momentum;

$$dQ = d\dot{m}(\omega r)r = (\rho U_2 2\pi r dr)(\omega r)r \quad (30)$$

Since $U_2 = U(1 - a)$, i.e., Eq (10) and $a' = \frac{\omega}{2\Omega}$ , Eq (25),

$$dQ = 4a'(1 - a)\frac{1}{2}\rho U\Omega r^2 2\pi r dr \quad (31)$$

The power generated in each element, $dP$ is

$$dP = \Omega dQ \quad (32)$$

Substituting (31) into (32) and using (29)

We arrive at

$$dP = \frac{1}{2}\rho A U^3 \left[\frac{8a'(1-a)\lambda_r^3}{\lambda^2} d\lambda_r\right] \quad (33)$$

where $A = \pi R^2$

Incremental contribution to the power coefficient ($dC_p$) from annual ring is

$$dC_p = \frac{dP}{\frac{1}{2}\rho A U^3} \quad (34)$$

Thus, total power coefficient ($C_p$) is given by

$$C_p = \frac{1}{\lambda^2}\int_0^\lambda a'(1 - a){\lambda_r}^3 d\lambda_r \quad (35)$$

From (28), we can express $a'$ in terms of $a$,

$$a' = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{{\lambda_r}^2}a(1-a)} \tag{36}$$

Integrand in (35) is

$$f(a) = (1-a)\left\{-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{{\lambda_r}^2}a(1-a)}\right\} \tag{37}$$

The maximum of $f(a)$ gives the maximum of $C_p$,

$\frac{df(a)}{da} = 0$ gives us

$${\lambda_r}^2 = \frac{(1-a)(4a-1)^2}{(1-3a)} \tag{38}$$

Substituting (38) into (28) for maximum power in each annual ring

$$a' = \frac{1-3a}{4a-1} \tag{39}$$

By taking derivatives of (38), we obtain

$$2\lambda_r d\lambda_r = \frac{6(4a-1)(1-2a)^2}{(1-3a)^2}da \tag{40}$$

Substituting (38)~(40) into (35), the maximum power coefficient, $C_{p,max}$ is obtained as

$$C_{p,max} = \frac{24}{\lambda^2}\int_{a_1}^{a_2}\left[\frac{(1-a)(1-2a)(1-4a)}{(1-3a)}\right]^2 da \tag{41}$$

where $a_1$corresponds to axial induction factor for $\lambda_r = 0$, $a_2$ to the axial induction factor for $\lambda_r = \lambda$.

From (38),

$$\lambda^2 = \frac{(1-a_2)(1-4a_2)^2}{(1-3a_2)} \tag{42}$$

Note in (42), $a_1 = 0.25,\ \lambda_r = 0$

Note in (42), $a_2 = \frac{1}{3},\ \lambda \to \infty$, thus $a_2 = \frac{1}{3}$ is the upper limit of axial induction factor

Integral in (41) is performed by setting[1] $x = (1-3a)$, then, $C_{p,max}$ is expressed as

$$C_{p,max} = \frac{8}{729\lambda^2}\left[\frac{64}{5}x^5 + 72x^4 + 124x^3 + 38x^2 - 63x - 12\ln x - \frac{4}{x}\right]_{x=(1-3a_2)}^{x=0.25} \tag{43}$$

[1] Eggleston, D. M. and Stoddard, F. S., 1987, Wind Turbine Engineering Design, Van Nostrand Reinhold, New York.