3.1 INTRODUCTION

When similar type of machines are available in multiple numbers and jobs can be scheduled over these machines simultaneously, parallel machines scheduling environment is at hand as shown in Figure 3.1 below.

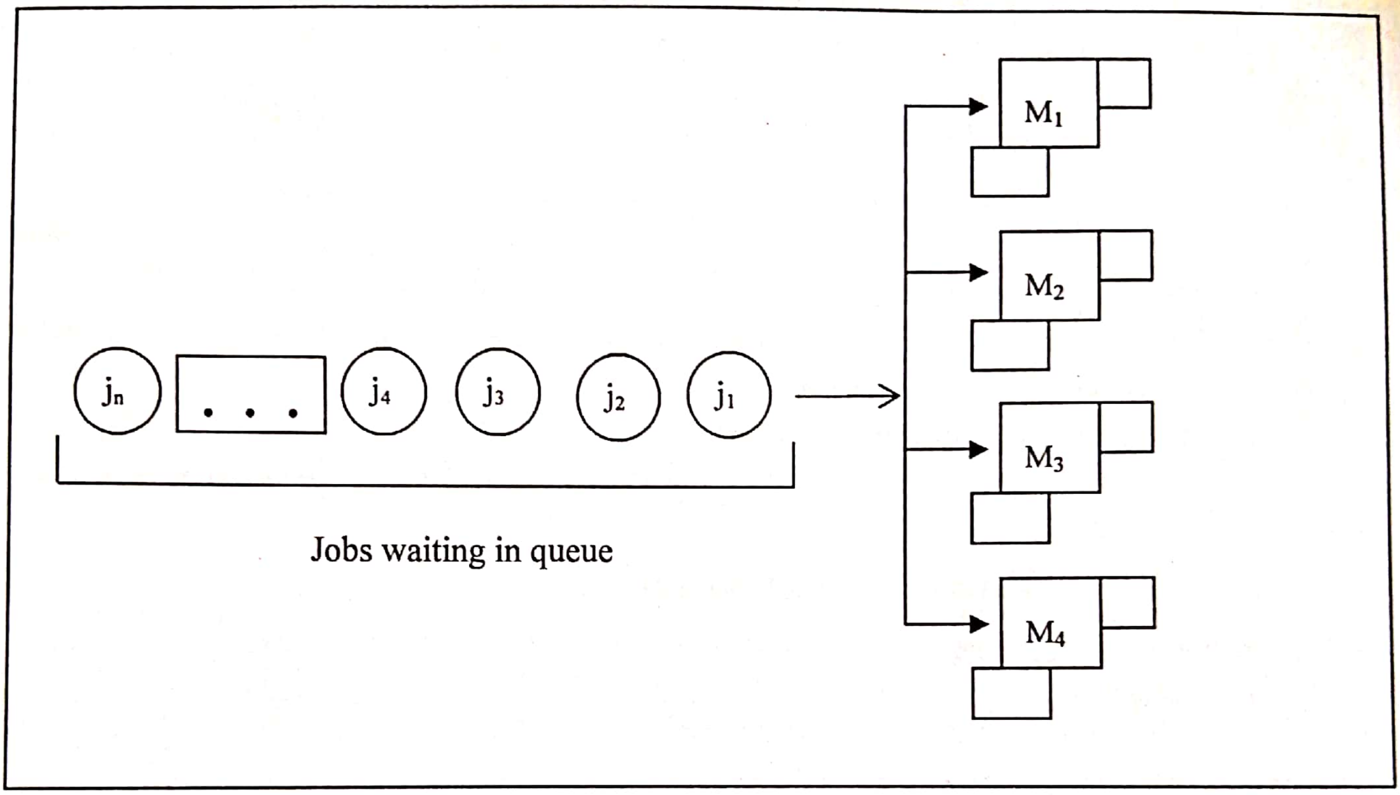

Figure 3.1 Four Parallel Machines with n-jobs.

The existence of parallel machines environment is common in real world flow shop and job shops systems. Knowledge of parallel machines modeling is useful to design the large-scale flexible flow shop and job shop systems

.2 MINIMIZATION OF MAKESPAN PROBLEM ($P_m \,||\, C_{max}$)

This problem deals with scheduling m parallel machines when the objective function is to minimize the makespan. A variety of heuristics are employed to determine a near-optimal schedule. Some of these heuristics include longest processing time first (LPT) rule and load- balancing heuristic.

3.1.1 Longest Processing Time first (LPT) Rule

A common heuristic used in parallel machines scheduling is the LPT rule. According to this heuristic, jobs are arranged in decreasing order of process times. The jobs having large values of process times are given high priority for scheduling on the parallel machines. The following relationship can be used to find how far the solution obtained by the LPT rule is far from an optimal solution.

$$\frac{C_{max}(LPT))}{C_{max}(OPT))} \le \frac{4}{3} - \frac{1}{3m}$$

Example 3.1

Using LPT rule, find the best schedule for jobs on the machines for the following $P_4 \| C_{max}$ problem.

Job (j)	1	2	3	4	5	6	7	8	9
p_j	7	7	6	6	5	5	4	4	4

Solution:

The LPT sequence is as follows: 1-2-3-4-5-6-7-8-9. From the LPT sequence, select job 1 to be scheduled on machine 1, then, select job 2 to be scheduled on machine 2, next, select job 3 to be scheduled on machine 3, and finally job 4 to be scheduled on machine 4. The partial schedule generated for each machine is presented in the following table

Table 3.1 Partial schedule for the four machines.

Machine (M)	Job assigned	Start Time S_j	Process Time p_{ij}	Completion Time (C_j)
M_1	1	0	7	7
M_2	2	0	7	7
M_3	3	0	6	6
M_4	4	0	6	6

From the LPT sequence, the unscheduled jobs are {5-6-7-8-9}. Since machines 3 and machine 4 are free and, available for next jobs at times 6, schedule job 5 and job 6 at these machines at time 6. The schedule of these jobs is presented in Table 2 as follows:

Table 3.2 Partial schedule for job set $\{j_5, j_6\}$.

Job (j)	Machine (M)	Start Time S_j	Process Time p_{ij}	Completion Time C_j
j_5	M_3	6	5	11
j_6	M_4	6	5	11

The next unscheduled jobs in the ordered set are job 7 and job 8. Machine 1 and machine 2 are free and available for next jobs at time 7. Schedule job 7 and job 8 at machine 1 and machine 2 at time 7 respectively. The schedule of these jobs is presented in Table 3 as follows:

Table 3.3 Partial schedule for job set $\{j_7, j_8\}$.

Job (j)	Machine (M)	Start Time S_j	Process Time p_{ij}	Completion Time C_j
j_7	M_1	7	4	11
j_8	M_2	7	4	11

The next unscheduled job in the ordered set is job 9. Machine 1, machine 2, machine 3, and machine 4 are all free for next job at time 11. Schedule job 9 on any machine; say at machine 1 at time 11. The complete schedule of all jobs on 4 parallel machines is given in Table 4 as follows:

Table 3.4 Complete schedule for job set $\{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9\}$.

Job (j)	Machine (M)	Start Time S_j	Process Time p_{ij}	Completion Time C_j
j_1	M_1	0	7	7
j_2	M_2	0	7	7
j_3	M_3	0	6	6
j_4	M_4	0	6	6
j_5	M_3	6	5	11
j_6	M_4	6	5	11
j_7	M_1	7	4	11
j_8	M_2	7	4	11
j_9	M_1	11	4	15

Gantt chart for $P_4 \| C_{max}$ schedule (shown in Table 3.4) is presented in Figure 3.2 as shown below;

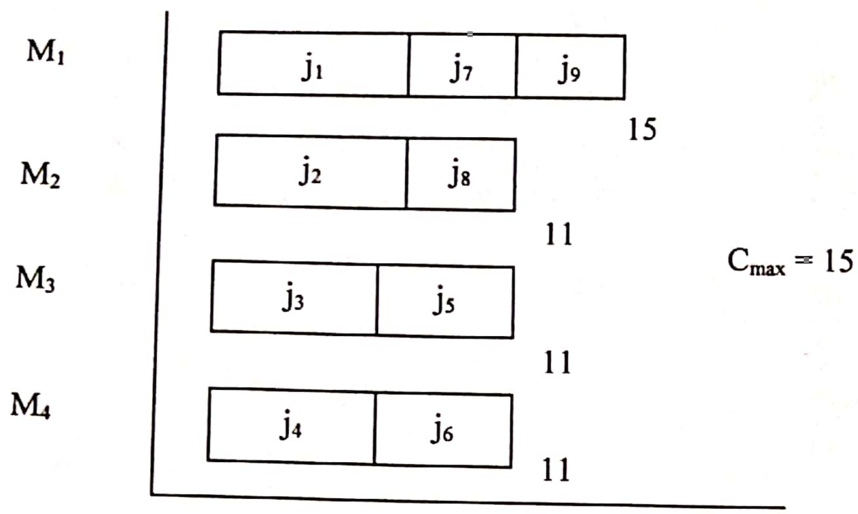

Figure 3.2 Gantt chart For P4|| C_{max} Problem. Note C_{max} = 15.

Example 3.2

Is the C_{max} value obtained in Example 3.1 optimal? If not, what is the optimal solution?

Solution:

For finding optimality of schedules, the following ratio between C_{max}(OPT) and C_{max}(LPT) is observed;

$$\frac{C_{max}(LPT))}{C_{max}(OPT))} \leq \frac{4}{3} - \frac{1}{3m}$$

Since, there are 4 machines, so m = 4. The ratio $\frac{C_{max}(LPT))}{C_{max}(OPT))} \leq 1.25$

The ratio of 1.25 indicates that LPT rule will give 25% more value of C_{max} in worst case as compared to C_{max} using optimal methodology. Hence, the optimal value of C_{max} in Example 3.1 should be 12, as calculated as follows:

Since, C_{max}(LPT) = 1.25 x C_{max}(OPT),
This implies that; C_{max}(OPT) = C_{max}(LPT)/1.25 = 15/1.25 = 12

The optimal value of C_{max} in Example 3.1 can be calculated by using *Load Balancing* heuristic as follows;

Let T_w = Total work content of all jobs in the problem. Then, tentative load per machine may be estimated by taking the ratio of T_w and m as follows:

$$\text{Load per machine} = \frac{T_w}{m}$$

For data presented in Example 3.1, $T_w = \sum_{j=1}^{9} p_j = 48$, and m = 4. Hence, tentative load per machine is 12. Table 3.5 presents combination of jobs for which average load per machine is 12. Optimal schedule is presented in Gantt chart (Figure 3.3).

Table 3.5 Optimal Schedule using Load Balancing Heuristic.

Machine	Jobs	Total Time
M_1	j_1, j_5	7 + 5 = 12
M_2	j_2, j_6	7 + 5 = 12
M_3	j_3, j_4	6 + 6 = 12
M_4	j_7, j_8, j_9	4 + 4 + 4 = 12

Gantt chart for $P_4 \| C_{max}$ schedule (shown in Table 3.5) is presented in Figure 3.3 as shown below;

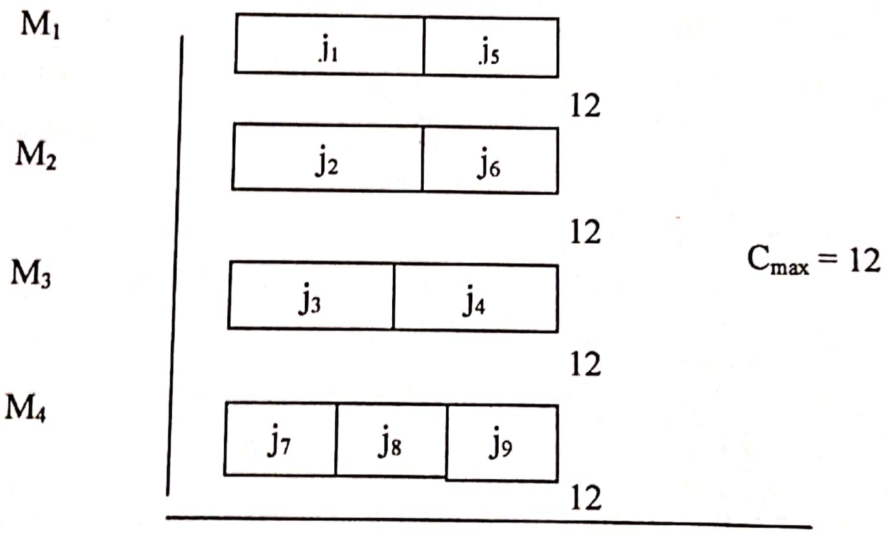

Figure 1.3 Gantt chart for optimal solution.