Lecture (18)

Nondivergent Barotropic Model

18.1 Barotropy

The barotropic model is of historical interest because it was used in the first successful attempt at NWP. In this model, the forecast equation involves only one dependent variable and is applied at a single pressure level in the vertical. In the barotropic atmosphere the density is a function of pressure alone; therefore, the density, temperature, and pressure surfaces are parallel; hence only one level needs to be forecasted. The thermal wind equation

$$\frac{\partial \vec{V}_g}{\partial \, lnP} = -\frac{R}{f}\vec{k} \times \nabla_{\mathrm{p}} T \quad , \qquad (18.1)$$

therefore becomes $\frac{\partial \vec{V}_g}{\partial \, lnP} = 0$, which states that the geostrophic wind is independent of height in a barotropic atmosphere. Here $\vec{V}_g$ is the geostrophic wind, P is air pressure, R is gas constant, f is the Coriolis parameter, k is the unit vector in z-axis, T is air temperature. Thus, barotropy provides a very strong constraint on the motions in a rotating fluid; the large-scale motion can depend only on horizontal position and time, not on height.

An atmosphere in which density depends on both the temperature and the pressure, $\rho = \rho(P, T)$, is referred to as a baroclinic atmosphere. In a baroclinic atmosphere the geostrophic wind is generally has vertical shear, and this shear is related to the horizontal temperature gradient by the thermal wind equation (18.1). Obviously, the baroclinic atmosphere is of primary importance in dynamic meteorology.

18.2 Dynamic of Barotropic Model

This model based on the conservation of absolute vorticity. The vorticity equation can be written in the form:

$$\frac{D}{Dt}(\xi + f) = -(\xi + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \quad (18.2)$$

where $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity. Equation (18.2) states that the rate of changing of the sum of the absolute vorticity following the motion is given by the sum of the three terms on the right, called the divergence term, the tilting or twisting term, and the solenoidal term, respectively.

The order of magnitude analysis proves that the second and the third terms on the right in this equation are sufficiently small so that they can be neglected for synoptic scale motion. Hence the vorticity equation may be rewritten as:

$$\frac{D}{Dt}(\xi + f) = -(\xi + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{18.3}$$

18.3 Vorticity in Barotropic Fluids

The continuity equation for a homogeneous incompressible fluid, simplifies to:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial w}{\partial z} \tag{18.4}$$

so that the vorticity equation (18.3) may be written as:

$$\frac{D}{Dt}(\xi + f) = (\xi + f)\left(\frac{\partial w}{\partial z}\right) \tag{18.5}$$

If the flow is purely horizontal (w=0), as is the case for barotropic flow in a fluid of constant depth, the divergence term vanishes in (18.5) and one obtains the barotropic vorticity equation:

$$\frac{D}{Dt}(\xi + f) = 0 \tag{18.6}$$

which states that absolute vorticity is conserved following the horizontal motion. More generally, absolute vorticity is conserved for any fluid layer in which the divergence of the horizontal wind vanishes, without the requirement that the flow be geostrophic. For horizontal motion that is nondivergent $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\right)$, the flow field can represented by a streamfunction $\psi(x, y)$ defined so that the velocity components are given as $u = -\frac{\partial \psi}{\partial y},\ v = +\frac{\partial \psi}{\partial x}$. The vorticity is then given by:

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \equiv \nabla^2 \psi \tag{18.7}$$

Thus, the velocity field and the vorticity can both be represented in terms of the variation of the single scalar field $\psi(x, y)$, and (18.6) can be written as a prognostic equation for vorticity in the form:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\vec{V}_\psi \cdot \nabla(\nabla^2\psi + f) \qquad (18.8)$$

where $\vec{V}_\psi \equiv \vec{k} \times \nabla\psi$ is a nondivergent horizontal wind. Equation (18.8) states that the local tendency of relative vorticity is given by the advection of absolute vorticity. This equation can be solved numerically to predict the evolution of streamfunction, and hence of the vorticity and wind fields. Since the flow in the mid-troposphere is often nearly nondivergent on the synoptic scale, (18.8) provides a surprisingly good model for short-term forecasts of the synoptic scale 500-hPa flow field. Equation (18.8) can be expressed as:

$$\frac{\partial}{\partial t}\nabla^2\psi = \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \beta\,\frac{\partial\psi}{\partial x} \qquad (18.9)$$

or simply,
$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi, \nabla^2\psi) - \beta\,\frac{\partial\psi}{\partial x} \qquad (18.10)$$

where $\beta = \frac{\partial f}{\partial y}$ and J is the Jacobian operator which is a common operator. The finite-difference form of this Jacobian can be written as:

$$JJ = \frac{1}{4h^2}[(\psi_{i+1,j} - \psi_{i-1,j}).(\xi_{i,j+1} - \xi_{i,j-1}) - (\psi_{i,j+1} - \psi_{i,j-1}).(\xi_{i+1,j} - \xi_{i-1,j}) \qquad (18.11)$$

The use of the nonlinear balance equation is of great usefulness to close the system:

$$\nabla^2 gz = \nabla \cdot f\nabla\psi + 2J\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) \qquad (18.12)$$

This equation defines the height field from the streamfunction, and together with (18.10) constitutes a closed system from the evolution of the barotropic nondivergent flow. Equation (18.12) represents actually a nonlinear reverse balance law since it determines the geopotential height from the wind field.

Reference:

Thaer O. Roomi, 2013: A basic regional-scale model for numerical weather prediction. PhD Thesis, Al-Mustansiryah Universiy, Iraq, Baghdad.