MORE EXAMPLES ABOUT THE ASYPTOTIC NOTATION AND COMPLEXITY

Example 1:

int sun = 0;	C_0
for (i=0, i<10, i++)	n
sum = sum +I ;	C_1

<u>What is the time complexity?</u>

Sol:

$C_0 + nC_1 =$ n time (Worst Case)

Example 2:

int z;	C_0
for (i=0; i<10, i++)	n
for (j=0; j<5+ j++)	n
z= i*j;	C_1

<u>What is the time complexity?</u>

$C_0 + n * n * C_1$

$C_0 + C_1 n^2 = n^2$ (Worst Case)

Example 3:

input f;	C_0
for (i=0' i<5, i++)	n
if (i=2)	C_1
print (i)	C_2

What is the time complexity?

Sol:

$C_0 + n(C_1 + C_2)$

$C_0 + nC_1 + nC_2 \longrightarrow$ time complexity = n

THE RULE:

1. (+, ÷ , × , *, - , if) 1 step
2. Loop, subroutine n steps
3. func (sum) 1 step, 1-n, 2-n, 3-sum, ….
4. Access memory 2 steps

Calculate the time for (FOR STATMENTS):

Case 1:

1. For (i=0, i<n, i++) n+1

$\underbrace{0\text{------------ } n-1}_{n}$

i< n

:

:

:

9 < 10

10 < 10

Case 2:

2. for (i=0; i<=n; i++) n+2

0 ------- n

n+1+1

Case 3:

for (i=1; i<n; i++) n

$\underbrace{1\text{------------}n-1}$

n-1+**1 (failure)**

Case 4:

for (i =1; i<=n; i++) n+1

$\underbrace{1\text{------------}n}_{n+1}$

Theory:

f(n)= am n^m + am -1 n^{m-1} + … + a1 n1 + a_0

f(n) = O (n^m)

RECURSION

Recursion means calling a function in itself. If a function invokes itself, then the phenomenon is referred to as recursion. However, in order to generate an answer, a terminating condition is must. In order to understand the concept, let us take an example. If the factorial of a number is to be calculated using the function fac(n) defined as follows:

fac(n) = n × fac(n-1)

and fac(1) = 1, and if the value of n is 5, then the process of calculating fac(5) can be explained with the help of Fig. 3.1. fac(1) is calculated and its value is used to calculate fac(2), which in turn is used for calculating fac(3). fac(3) helps to calculate fac(4) and finally, fac(4) is used to calculate fac(5). As is evident from Fig. 3.1, recursion uses the principle of last in first out and hence requires a stack. One can also see that had there been no fac(1), the evaluation would not have been possible. This was the reason for stating that recursion requires a terminating condition also.

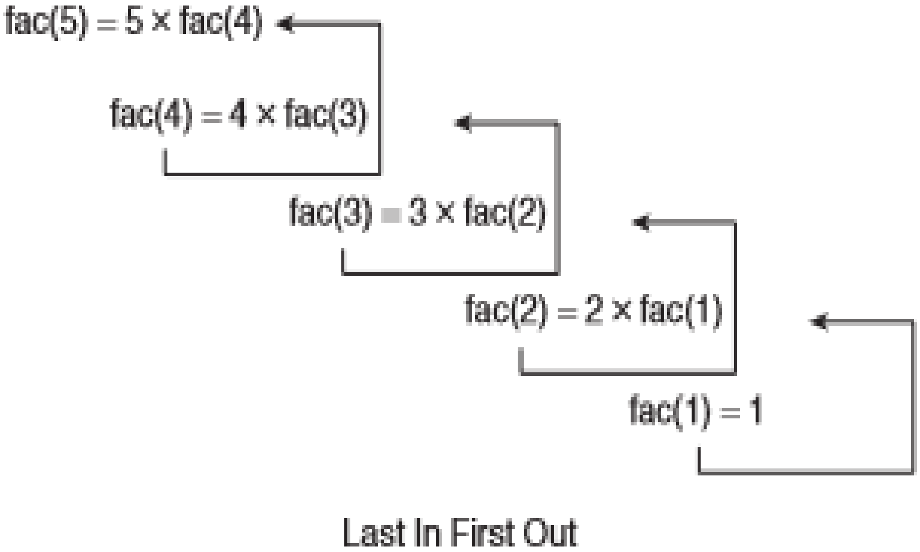

Figure 3.1 Calculation of factorial of 5

DERIVING AN EXPLICIT FORMULA FROM RECURRENCE FORMULA

Though a recurrence formula gives an idea of how a particular term is related to the previous or the following term, it does not help us to directly find a particular term without having gone through all the intervening terms. For that, we need an explicit formula. There are three methods for finding an explicit formula from a recurrence relation. They are as follows:

- Substitution
- Generating functions
- Tree method

The choice of the method, however, is a precarious issue. There is no thumb rule to determine which method to be used for a particular relation.

1. SUBSTITUTION METHOD

The solution of a recurrence equation by substitution requires a previous instance of the formula to be substituted in the given equation. The process is continued till we are able to reach to the initial condition. Illustration 3.1 gives an example of the method.

Illustration 1: Solve the following recurrence relation by substitution:

$$a_n = 2 \times a_{n-1} + 3, \quad n \geq 2$$

$$a_1 = 2, \quad n = 2$$

Solution Since, $a_n = 2 \times a_{n-1} + 3$, $n \geq 2$, therefore

$$a_{n-1} = 2 \times a_{n-2} + 3, \quad n \geq 2 \qquad (3.1)$$

Substituting the value of a_{n-1}, we get

$$a_n = 2 \times \left((2 \times a)_{n-2} + 3\right) + 3 \qquad (3.2)$$

which is same as, $a_n = 4 \times a_{n-2} + 2 \times 3 + 3$

From the given equation, it can be inferred that

$$a_{n-2} = 2 \times a_{n-3} + 3 \qquad (3.3)$$

By substituting Eq. (3.3) in Eq. (3.2), we get

$$a_n = 2 \times \left((2 \times (2 \times a)_{n-3} + 3) + 3\right) + 3, \text{ that is,}$$

$$a_n = 2^r \times a_{n-r} + 3 \times \left(1 + 2 + \cdots + 2^{r-1}\right)$$

or $$a_n = 2^r \times a_{n-r} + 3 \times \left(2^r - 1\right) \qquad (3.4)$$

Putting $n - r = 2$ or $r = (n-2)$

we get $$a_n = 2^{n-2} \times a_2 + 3 \times \left(2^{n-2} - 1\right)$$

Since, $a_2 = 2$

Therefore, $$a_n = 2^{n-2} \times 2 + 3 \times \left(2^{n-2} - 1\right)$$

This implies, $$a_n = 2^{n-1} + 3 \times \left(2^{n-2} - 1\right)$$

or, $$a_n = \frac{5}{2} 2^{n-1} - 3$$

2. SOLVING LINEAR RECURRENCE EQUATION

A linear recurrence relation of order 'r' with constant coefficients is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r}$$

where $c_r \neq 0$.

For example,

$a_n = 3a_{n-1}$ is a recurrence relation of order 1.

$a_n = a_{n-1} + a_{n-2}$ is also a recurrence relation, which depicts the Fibonacci series, of order 2.

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$ is a recurrence relation of order 3.

The first step in solving a recursive relation is to form its characteristic equation. A characteristic equation is a polynomial equation formed by retaining the constants of the given equation and by replacing with the powers of s as shown in the following examples. As a matter of fact, the answer depends on the solution of the equation. So, it does not really make a difference, if one opts for other variables, except for s.

Examples of characteristic equations:

- For the equation $a_n = c_1 a_{n-1}$
  The characteristic equation would be $s = c_1 s^0$
- For the equation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
  The characteristic equation would be $s^2 = c_1 s^1 + c_2 s^0$
- For the equation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$
  Then characteristic equation would be $s^3 = c_1 s^2 + c_2 s^1 + c_3 s^0$

The next step is to solve the characteristic equation. We must be familiar with the solution of a quadratic or a cubic equation. The contentious point is, therefore, to be able to find α_n from the roots of the characteristic equation. The following rules would help us to do so. Solving the characteristic equation, we get roots $\alpha_1, \alpha_2, \alpha_3, \ldots$.

- If $\alpha_1, \alpha_2, \alpha_3, \ldots$ are all distinct, then the solution is

$$a_n = c_1(\alpha_1)^n + c_2(\alpha_2)^n + c_3(\alpha_3)^n + \cdots$$

- If two roots α_1 and α_2 are same, the solution is of the form

$$a_n = (c_1 + nc_2)(\alpha_1)^n$$

- If the characteristic equation has 3 roots and all are equal, then

$$a_n = (c_1 + nc_2 + n^2 c_3)(\alpha_1)^n$$

The following illustrations would help us to understand the above concepts.

Illustration: Solve the following recurrence relation:

$$a_n = a_{n-1} + 6a_{n-2}$$

$$a_1 = 1,\ a_0 = 2$$

Solution For $a_n = a_{n-1} + 6a_{n-2}$, the characteristic equation would be

$$s^2 = s + 6s^0$$

By solving, we get $\quad s^2 - s - 6s^0 = 0$

or $\quad s^2 - 3s + 2s - 6 = 0$

or $\quad (s-3)(s+2) = 0$

Hence $\quad s = 3, -2$

So $\quad a_n = c_1(\alpha_1)^n + c_2(\alpha_2)^n = c_1(-2)^n + c_2(3)^n \qquad (3.9)$

Putting $n = 0$ in Eq. (3.9), we get

$$a_0 = c_1 + c_2 = 2$$

Putting $n = 1$ in Eq. (3.9), we get

$$a_1 = -2c_1 + 3c_2 = 1$$

i.e. $\quad c_1 + c_2 = 2 \qquad (3.10)$

or $\quad -2c_1 + 3c_2 = 1 \qquad (3.11)$

Solving the above two equations, we get $c_1 = c_2 = 1$

Putting the values in Eq. (3.9), we get

$$a_n = (-2)^n + (3)^n$$

3. GENERATING FUNCTIONS

The method of solving a recurrence relation is using generating functions. To be able to solve a recurrence relation via a generating function, let us first of all learn to form a generating function of a recurrence relation.

An infinite series

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$$

is called generating function of numeric function $(a_0, a_1, a_2, \ldots, a_n)$.

Case 1 If $(a_0 = a_1 = a_2, \ldots, = a_n = 1)$
Then, the generating function becomes

$$1 + z + z^2 + z^3 \cdots + z^n + \cdots$$

Please note that the above series is a GP and the sum $= \dfrac{a}{1-r} = \dfrac{1}{1-z}$

Hence, $A_{(z)} = \dfrac{1}{(1-z)}$

Case 2 For $2^0 + 2^1 z + 2^2 z^2 + 2^3 z^3 \cdots + 2^n z^n + \cdots$

$$A_{(z)} = \frac{1}{(1-2z)}$$

Generalization:

For $$a^0 + a^1 z + a^2 z^2 + a^3 z^3 \cdots + a^n z^n + \cdots$$

$$A_{(z)} = \frac{1}{(1-az)}$$

Illustration1: Find generating function for

$$a_n = 2.3^n + 4.5^n + 6.8^n$$

Solution Since for a^n, the generating function is

$$\frac{1}{(1-az)}$$

Therefore, for 3^n it becomes $\dfrac{1}{(1-3z)}$

for 5^n it becomes $\dfrac{1}{(1-5z)}$, and

for 8^n it becomes $\dfrac{1}{(1-8z)}$

The generating function for the given equation is

$$A_{(z)} = 2\frac{1}{(1-3z)} + 4\frac{1}{(1-5z)} + 6\frac{1}{(1-8z)}$$

Illustration2: Find generating function for

$$a_n = 3^{n+4}$$

Solution

$$a_n = 3^{n+4}$$
$$= 3^n \cdot 3^4$$
$$= 3^n \cdot 81$$

The generating function for $3^n = \frac{1}{1-3z}$

Therefore,

$$A_{(z)} = \frac{81}{1-3z}$$

Having seen the formation of a generating function for a recurrence relation, let us now see the method for finding the solution. In the following illustration, the value of A(z) is given and the recurrence relation is to be solved.

Points to Remember

- Recursion uses the principle of last in first out and hence requires a stack.
- In the Fibonacci series, the *n*th term can be found by taking the sum of the $(n-1)$th and $(n-2)$th term.
- Substitution, generating functions, and tree method are some of the methods to find the explicit formula for a recurrence equation.
- For the equation $a_n = c_1 a_{n-1}$, the characteristic equation would be $s = c_1 s^0$; for the equation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, the characteristic equation would be $s^2 = c_1 s^1 + c_2 s^0$; for the equation