$\mathbb{N}$

$\mathbb{N} = \{0, 1, 2, \ldots\}$

$0 = \emptyset$

$1 = \{\emptyset\}$

$2 = \{\emptyset, \{\emptyset\}\}$

$\vdots$

$0^+ = 1$

$1^+ = 2$

$2^+ = 3$

$\vdots$

$0^+ = 1 = 0 + 1$

$1^+ = 2 = 1 + 1$

$2^+ = 3 = 2 + 1$

$\vdots$

$n^+ = n + 1$

Theorems

تدريبات ٤

1. There is a set whose members are exactly the natural numbers.

2. $\mathbb{N}$ is a successor set, and is a subset of every other successor set.

3. Every successor subset of $\mathbb{N}$ coincides with $\mathbb{N}$.

Peano's Postulates

① $0 \in \mathbb{N}$ — proof ?

0 is a natural number

② $+$ successor operation

$$n \to n^+ = n+1$$

$$+ : \mathbb{N} \to \mathbb{N}$$

$$(+) S : \mathbb{N} \to \mathbb{N}$$

$$S(n) = n^+ = n+1$$

③ 0 is the first natural number.

④ $+ : \mathbb{N} \to \mathbb{N}$ $\overset{1-1}{\text{injective}}$

$$S(n) = S(m) \Rightarrow n = m \ ?$$

$$n^+ = m^+ \Rightarrow n = m$$

4

$\boxed{n^+ = n+1}$ $\qquad 1^+ = 2 = 1+1$

$\boxed{n = m?}$

$S(n) = S(m) \Rightarrow n^+ = m^+$

$\Rightarrow n\cancel{+1} = m\cancel{+1}$

$\Rightarrow n = m$

S , $+$ Successor operation

is 1-1

$$\boxed{a^+ = b^+ \Rightarrow a = b}$$

$+(a) = a^+$

$+(b) = b^+$ $\qquad$ 1-1

⑤

Principle of Induction:- P_5

Every successor subset of $\mathbb{N}$ concides with $\mathbb{N}$.

$$\left.\begin{array}{l} A \subseteq \mathbb{N} \\ \downarrow \\ \text{Successor set} \end{array}\right\} \Longrightarrow \underline{\underline{A = \mathbb{N}}}$$

$$A \subseteq \mathbb{N}$$

$$\downarrow$$

$$\left.\begin{array}{l} 0 \in A \\ k \in A \Longrightarrow k^+ \in A \end{array}\right\} \Longrightarrow A \text{ Successor set} \Longrightarrow A = \mathbb{N}$$

Addition + on $\mathbb{N}$

Let $a, b \in \mathbb{N}$.

$+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$+(a,b) = a+b = \begin{cases} a+0 = a & \text{if } b=0 \\ a+c^{+} = (a+c)^{+} & \text{if } b \neq 0 \end{cases}$$

Where $b = c^{+}$

Recursion theorem

Let A is a set, $a \in A$,

$F : A \longrightarrow A$. There exists a unique function $h : \mathbb{N} \rightarrow A$ s.t.

$$h(0) = a$$

$$h(n^{+}) = F(h(n)), \quad n \in \mathbb{N}$$

$F : + : \mathbb{N} \longrightarrow \mathbb{N}$

$h : A_m : \mathbb{N} \longrightarrow \mathbb{N}$

$A_m(n^+)$

$= A_m(0) = m$

$= A_m(n^+) = F(A_m(n))$

$= (A_m(n))^+$

$h = A_m$, $F = +$ } recursion theorem

Definition binary operation

$+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$m + n = A_m(n)$

Theorem

$m \in \mathbb{N}$

$$\boxed{\begin{aligned} m + 0 &= m \\ m + n^+ &= (m+n)^+ \end{aligned}}$$

$$a+b = \begin{cases} a & b = 0 \\ a+c^{+} = (a+c)^{+} & b \neq 0 \\ & b = c^{+} \end{cases}$$

$$1+1 = 1+0^{+} = (1+0)^{+}$$
$$= 1^{+}$$
$$= 2$$

$$1+2 = 1+1^{+} = (1+1)^{+}$$
$$= 2^{+}$$
$$= 3$$

Let $a, b \in \mathbb{N}$.

$\cdot : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$\boxed{\cdot(a,b) = a\cdot b = \begin{cases} a\cdot 0 = 0 & \text{if } b = 0 \\ a\cdot c^+ = a + a\cdot c & b \neq 0 \\ & b = c^+ \end{cases}}$$

$$\begin{aligned} 3\cdot 2 &= 3\cdot 1^+ \\ &= 3 + 3\cdot 1 \\ &= 3+3 \\ &= 3+2^+ \\ &= (3+2)^+ \\ &= 5^+ \\ &= 6 \end{aligned}$$

$0^+ = 1$

$1^+ = 2$

$2^+ = 3$

$3^+ = 4$

$\vdots$

$\cancel{2+3}$

$$\begin{aligned} 3+2 &= 3+1^+ \\ &= (3+1)^+ \\ &= 4^+ \end{aligned}$$

$$\begin{aligned} 3+1 &= 1+2^+ \\ &= (1+2)^+ \\ &= 3^+ \end{aligned}$$

$$\begin{aligned} 1+2 &= 1+1^+ \\ &= (1+1)^+ = 2^+ \end{aligned}$$

$$\begin{aligned} 1+1 &= 1+0^+ \\ &= (1+0)^+ = 1^+ = 2 \end{aligned}$$

Theorem 2.1.10

(11)

$n^+ = n+1$

$n = n \cdot 1$

$0 \cdot n = n$

$0 + n = n$

$m, n, c \in \mathbb{N}$

? ② $(m+n)+c = n+(m+c) \quad \forall\, m, n, c \in \mathbb{N}$

using principle of induction :-

Let

$L_{mn} = \{ p \in \mathbb{N} : m+(n+p) = (m+n)+p \}$

$\downarrow \; L_{mn} \subseteq \mathbb{N}$

L_{mn} is a successor set

$\downarrow$

$0 \in \mathbb{N}, \quad m+(n+0) = m+n$

$= (m+n)+0$

$\therefore \; 0 \in L_{mn}$

$$p \in L_{mn} \Rightarrow p^+ \in L_{mn} \ ?$$

$$m+(n+p) = (m+n)+p$$

$$m+(n+p^+) \overset{?}{=} (m+n)+p^+$$

$$(m+n)+p^+ = \big((m+n)+p\big)^+$$

$$= \big(m+(n+p)\big)^+$$

$$= m+(n+p)^+$$

$$= m+(n+p^+)$$

$$\therefore p^+ \in L_{mn}, \ \forall p \in L_{mn}.$$

$\therefore$ L_{mn} Successor Set

$\therefore$ $L_{mn} = \mathbb{N}$

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③ $\boxed{m+n = n+m \ ?}$

L_{mn} $\{n \in \mathbb{N} : m+n = n+m\}$

$\cap$

$\mathbb{N}$

Successor set.

let $A = \{ n \in \mathbb{N} : \underline{0+n = n} \} = \mathbb{N}$

$0 \in A$

$k \in A \longrightarrow k^+ \in A$?

$0 + k^+ = (0+k)^+ \ , \ k \in A$

$= k^+$

$\therefore k^+ \in A$

Let $B = \{ n \in \mathbb{N} : m^+ + n = (m+n)^+ \}$

B successor set ?

$0 \in B$

$k \in B \overset{?}{\Longrightarrow} k^+ \in B$

$$m^+ + k^+ = (m^+ + k)^+$$
$$= ((m+k)^+)^+.$$
$$= (m+k^+)^+$$

$\therefore k^+ \in B$

$L_{mn} = \{ m \in \mathbb{N} : m+n = n+m \}$

$0 \in L_{mn}$?

$\underline{0+n = n = n+0}$

$\therefore 0 \in L_{mn}$

let $k \in L_{mn} \longrightarrow k^+ \in L_{mn}$?

$$k^+ + n = (k+n)^+$$

$$= (n+k)^+$$

$$= n + k^+ .$$

$\therefore\ k^+ \in L_{mn}$.

$\therefore\ L_{mn}$ Successor Set ,

$L_{mn} = \mathbb{N}$.

4. $m\cdot(n+p) = m\cdot n + m\cdot p$

Let

$A = \{ p \in \mathbb{N} : m\cdot(n+p) = m\cdot n + m\cdot p \}$

$0 \in A$?

$m\cdot(n+0) = m\cdot n$

$= m\cdot n + 0$

$= m\cdot n + m\cdot 0$

Let $k \in A$. Then

$m\cdot(n+k^+) = m\cdot(n+k)^+$

$= m\cdot(n+k) + m$

$= (m\cdot n + m\cdot k) + m$

$= m\cdot n + (m\cdot k + m)$

$= m\cdot n + m\cdot k^+$

$\therefore k^+ \in A$, A is a successor set.

5

④ $m\cdot(n\cdot p) = (m\cdot n)\cdot p$

Consider m and n in $\mathbb{N}$ and let

$$A = \{ p \in \mathbb{N} : m\cdot(n\cdot p) = (m\cdot n)\cdot p \}$$

To check $0 \in A$,

$m\cdot(n\cdot 0) = m\cdot 0 = 0$

$(m\cdot n)\cdot 0 = 0$

Suppose $k \in A$. Then

$$m\cdot(n\cdot k^+) = m\cdot(n\cdot k + n)$$

$$= m\cdot(n\cdot k) + (m\cdot n)$$

$$= (m\cdot n)\cdot k + m\cdot n$$

$$= (m\cdot n)\cdot k^+$$

$\therefore k^+ \in A$ and hence A is a successor set.