

Chapter - 5 - System of Numbers

Consider the following elements :-

$$\emptyset$$

$$\{\emptyset\}$$

$$\{\emptyset, \{\emptyset\}\}$$

$$\{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$$

$$\vdots$$

and we introduce the usual numerical symbols $0, 1, 2, 3, \dots$ for these various sets. The symbol $n = 0, 1, 2, 3, \dots$ therefore represents the set above with n elements.

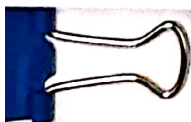
It follows from definition that

$$0 = \{\emptyset\}$$

$$1 = \{0\}$$

$$2 = \{0, 1\}$$

$$3 = \{0, 1, 2\}$$

$$\vdots$$


Definition

The successor of x is $x \cup \{x\}$ and is denoted by x^+

Thus,

$$0^+ = 0 \cup \{0\}$$

$$= \emptyset \cup \{\emptyset\}$$

$$= \{\emptyset\} = 1$$

$$1^+ = 1 \cup \{1\}$$

$$= \{\emptyset\} \cup \{\{\emptyset\}\}$$

$$= \{\emptyset, \{\emptyset\}\} = 2 = \{0, 1\}$$

$$2^+ = 2 \cup \{2\}$$

$$= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\}$$

$$= \{\emptyset, \{\emptyset\}, \{\{\emptyset, \{\emptyset\}\}\}\}$$

$$= 3$$

$$n^+ = n+1$$

Definition

Let A be a set, we call A a successor set if

- ① $0 \in A$
- ② $x^+ = x \cup \{x\} \in A$ whenever $x \in A$.

Notes :-

- ① Any successor set should contain the numbers $0, 1, 2, \dots, \infty$ ✗
- ② Collection of all successor sets is not empty ✗
- ③ Intersection of any non-empty collection of successor sets is also successor set.

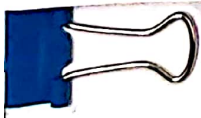
Definition :-

Intersection of all successor sets is called the set of natural numbers and is denoted by \mathbb{N} .

Each element of \mathbb{N} is called natural element.

Peano Postulates :-

- ① $0 \in \mathbb{N}$
- ② If $a \in \mathbb{N}$, then $a^+ \in \mathbb{N}$
- ③ $a \neq a^+ \in \mathbb{N}$ for every natural number a
- ④ If $a^+ = b^+$, then $a = b$ for any natural numbers a, b .
- ⑤ If X is a successor subset of \mathbb{N} , then $X = \mathbb{N}$



Definition

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- ① $0 \in A$
- ② $x^+ = x \cup \{x\} \in A$ whenever $x \in A$.

Notes :-

- ① Any successor set should contains the numbers $0, 1, 2, \dots, n$ * مطلوب برهان
- ② Collection of all successor sets is not empty * مطلوب برهان
- ③ Intersection of any non empty collection of successor sets is also successor set. ✓

Definition :-

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- ⑤ If X is a successor subset of \mathbb{N} , then $X = \mathbb{N}$ مطلوب

The intersection of any non-empty collection of successor sets is a successor set.

Let $B = \bigcap (\text{successor sets})$

$\therefore 0 \in B$ since $0 \in$ each successor set.

Let $z \in B$, then $z \in \bigcap (\text{Successor sets})$ and $z \in$ every successor set

then $z^+ \in$ every successor set and $z^+ \in \bigcap (\text{Successor sets})$

then $z \in B^+$ and

B is a successor set.

Peano Postulates

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clear from Theorem $\textcircled{*}$, below

Theorem (x)

There is a smallest successor set.
 Means there is a successor set which is a subset of all successor sets. This smallest successor set is of course the intersection of all successor sets.

Proof

The intersection of all successor sets is a successor set.

Now to prove that the intersection is the smallest successor set, we need only to prove that the intersection is a subset of every successor set.

Let $S = \bigcap \text{all successor sets}$

and let $x \in S$, and let t be any successor set

$\{x, t\}$ is a set, and $x \cap t$ is a set and it is a successor set

$$S \subseteq x \cap t$$

Hence, S is the smallest successor set.

