Lecture One

Mathematical Basic Concepts

3. Arithmetic Functions

Arithmetic (or number theoretic) functions are the most fundamental functions in mathematics and computer science; for example, the computable functions studied in mathematical logic and computer science are actually arithmetic functions. In this section we shall study some basic arithmetic functions that are useful in number theory.

Definition (3.1): A **function** f is a rule that assigns to each element in a set D (called **Domain** of f) one and only one element in a set B. the set of images called the **range** (R) of f [7].

Definition (3.2):

1. The function f has the property of being "**one-to-one**" (or "**injective**") if no two elements in D are mapped into the same element in R.
2. The function f has the property of being "**onto**" (or "**surjective**") if the range R of f is all of B (R=B).

Definition (3.3): Given functions f and g, the **composition** of f with g, denoted by $f \circ g$ is the function by:

$(f \circ g)(x)=f(g(x))$

The domain of $f \circ g$ is defined to consists of all x in the domain of g for which g(x) is in the domain of f.

Definition (3.4): A function f is called an **arithmetic function** or a **number theoretic** function if it assigns to each positive integer n a unique real or complex number $f(n)$. Typically, an arithmetic function is a real-valued function whose domain is the set of positive integer.

Example (3.1): the equation $\sqrt{n}$, $n \in \mathbb{N}$, defines an arithmetic function f which assigns the real number $\sqrt{n}$ to each positive integer.

Definition (3.5): A real function defined on the positive integers is said to be **multiplicative** if:

$f(m)f(n)=f(mn)$, $\forall m,n \in \mathbb{N}$ with $\gcd(m,n)=1$.

Definition (3.6): Let n be a positive integer. **Euler's (totient)** Φ-function, $\Phi(n)$ defined to be the number of positive integer k less than n which are relatively prime to n:

$$\Phi(n) = \sum_{\substack{1 \le k < n \\ \gcd(k,n)=1}} 1$$

Example(3.2): By definition (3.6) we have:

n	1	2	3	4	5	6	7	8	9	10	100	101	102	103
Φ(n)	1	1	2	2	4	2	6	4	6	4	40	100	32	102

Theorem (3.1): Let n be a positive integer, then

1. $\Phi(n)$ is multiplicative i.e. $\Phi(mn)= \Phi(m)\,\Phi(n)$.
2. if n is prime, say p, then $\Phi(p)=p-1$, and if n is prime power p^{α}, then $\Phi(p^{\alpha})= p^{\alpha}- p^{\alpha-1}= p^{\alpha-1}(p-1)$.
3. if n is composite and has the standard prime factorization form, then

$$\Phi(n)=p_1^{\alpha_1-1}(p_1-1)\cdot p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1)=\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1).$$

4. $\Phi(n)=(p-1)(q-1)$ if $n=pq$, where p and q are prime numbers.

Definition (3.7): Let x be a positive real number ≥ 1, then $\pi(x)$ is defined as follows:

$$\pi(x)=\sum_{\substack{p\leq x \\ p \text{ prime}}} 1 .$$

$\pi(x)$ is called the **prime counting** function (or the **prime distribution** function).

Example (3.3): $\pi(1)=0$, $\pi(2)=2$, $\pi(10)=4$, $\pi(20)=8$, $\pi(30)=10$, $\pi(40)=12$, $\pi(50)=15$, $\pi(75)=21$, $\pi(100)=25$.