Lecture One

Mathematical Basic Concepts

10. Linear Equations Systems and Matrices

10.1 Linear Equations

Let F be field, let $a_1,a_2,\ldots,a_n,b\in F$ and $x_1,x_2,\ldots,x_n$ be variables (unknowns), then the combination equation:

$a_1x_1+a_2x_2+\ldots+a_nx_n=b$

called ***Linear Equation***, $a_1,a_2,\ldots,a_n$ are ***coefficients*** and b be the ***absolute value***.

A collection of linear equations is:

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2$

$\vdots$

$a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=b_m$

called ***m-system of linear equations*** with n variables.

If $(b_1=b_2=\ldots=b_m=0)$, then the system called ***Homogeneous Linear Equations***, otherwise its called ***Non Homogeneous Linear Equations***.

The values x_1, x_2, …, x_n which are satisfied the system of linear equations is called ***solution***.

Example (10.1): For the following system:

$2x_1+3x_2+8x_3+x_4= 6$

$x_1+x_2+3x_3-x_4= 2$

$3x_1-4x_2+8x_3-x_4=5$

(-1,0,1,0) is a solution and (-3,6,-1,2) is another solution.

10.2 Matrices

In example (10.1), the coefficients of the linear equations can be written as follows:

$$\begin{bmatrix} 2 & 3 & 8 & 1 \\ 1 & 1 & 3 & -1 \\ 3 & -4 & 8 & -1 \end{bmatrix}$$

This model called a ***Matrix***.

The matrix is rectangular arrangement with orthogonal rows and columns, it can be put in parentheses ( ) or [ ].

Every equation in the linear equations system is row in the matrix while, the column is same variable with different coefficient.

The general form of the matrix is:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The matrix A consists of m rows with n columns so it can be denoted by $(a_{ij})_{mxn}$, or can be denoted by capital letters A,B,C…

10.3 Types of Matrices

The most important kinds of matrices are:

1. **Square matrix**: the matrix is called square matrix if m=n.
2. **Zero matrix**: It's the matrix which all it elements are 0's, and it denoted by O.

3. **Identity matrix**: its square matrix which all elements are 0's, accept its 1's on the main diagonal, and it denoted by I.
4. **Transpose Matrix**: change the ever row of the matrix to column, and it denoted by A^t.
5. **Triangular matrix**: the square matrix which all elements under the main diagonal are 0's called up-triangular matrix, while its called down-triangular matrix which all elements above the main diagonal are 0's.
6. **Diagonal matrix**: it's the matrix which all elements under and above the main diagonal are 0's.

10.4 Operations on Matrices

We will discuss three important operations on matrices. Two matrices are equal when they were from the same degree and the corresponding elements are equal.

10.4.1 Multiply by Scalar

Multiply the matrix by scalar done when all its elements are multiplied by the same scalar.

<u>Example (10.2)</u>

If the scalar is 2 then:

$$2.\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 6 \\ 0 & -8 \end{bmatrix}$$

10.4.2 Addition of Matrices

We can add only the matrices from the same degree, this operation done when adding the corresponding elements of the two matrices.

$$(a_{ij})_{mxn} + (b_{ij})_{mxn} = (a_{ij}+b_{ij})_{mxn}$$

Example (10.3)

$$\begin{bmatrix} 2 & 0 & 5 \\ -1 & 3 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 4 & -3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 4 & 2 \end{bmatrix}$$

10.4.3 Matrices Multiplying

We can multiply two matrices if the number of columns of the first matrix equals the number of rows of the second matrix, then the degree of the result matrix is equal to row of the first matrix by the columns of the second matrix, the general form id:

$$(a_{ij})_{mxk} \times (b_{ij})_{kxn} = \left(\sum_{t=1}^{k} a_{it} \cdot b_{tj} \right)_{mxn}$$

Example (10.4):

$$\begin{bmatrix} 2 & 0 & 5 \\ -1 & 3 & -2 \end{bmatrix}_{2\times 3} * \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 10 & 13 \\ -1 & -4 \end{bmatrix}_{2\times 2}$$

10.5 Determinants

It's a function with domain is the set of all square matrices with range is the field F. the value of this function is called the determinant of the matrix and its denoted by |A|. The calculation of the matrix determinant done with respect to its degree, which as follows:

1. 1×1 matrix: if A=[a], then |A|=a.

2. 2×2 matrix: if $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|=a*d-c*b$.

3. 3×3 matrix: if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$$|A|=a_{11}*a_{22}*a_{33}+ a_{12}*a_{23}*a_{31}+ a_{13}*a_{21}*a_{32}-a_{13}*a_{22}*a_{31}-a_{11}*a_{23}*a_{32}+ a_{12}*a_{21}*a_{33}$$

10.6 Inverse of Matrices

The square matrix B will be called the inverse of the square matrix A if:

$A \times B = B \times A = I$

And it's denoted by A^{-1}. There are many methods to find the inverse of the matrix like, adjacent matrix method, elementary matrix, Jordan method triangular method…, etc.

Theorem (10.1)

Let A be square matrix, then A will be invertible matrix if and only if its determinant not equal zero.

10.7 Numerical Solutions of the Linear Equations Systems

Let's have the following linear equations system:

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2$

$\vdots$

$a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=b_m$

This system consists of m equations with n variables. If we use the matrix notation then the above system will be: AX = B s.t.

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is the coefficient matrix,

$$X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ is the unknown (variables) matrix, and $$B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$ is the absolute value matrix.

The augment matrix is the matrix A beside it the column B.

$$[A|B]=\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The solution of the linear equations systems means find the values of the unknowns $x_1,x_2,\ldots,x_n$ which satisfy all equations of the system.

Definitions (10.1)

The zero solution is the solution of the linear system if all the valus of matrix X equal 0, s.t.

$x_1 = x_2 = x_3 = \ldots = x_n = 0$

Definitions (10.2)

The square matrix A called singular if and only if $|A|=0$.

Theorem (10.2)

Let A be square matrix of degree n, then the following relation are equivalent:

1. The homogenous system AX = 0 has zero solution only.
2. The system AX = B has unique solution for every different column B.
3. The matrix A has inverse.

10.8 Matrices Solving Methods

There are many methods for solving the matrices, we will describe some of them.

10.8.1 Cramer Rule

<u>Theorem (10.3)</u>

If A is the coefficient matrix of linear equations system consists of n variables, and $|A| \neq 0$, then the solution is:

$x_1 = D_1/D,\ x_2 = D_2/D,\ \dots,\ x_n = D_n/D,$

where $D=|A|$, and $D_i=|A_i|$, A_i is the matrix A when change the column B with column i, s.t. $1 \le i \le n$.

<u>Example (10.5)</u>

$3x_1 - 2x_2 = 6$

$2x_1 + x_2 = 0.5$

AX = B

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 6 \\ 0.5 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$|A| = 7$

$$x_1 = \frac{\begin{bmatrix} 6 & -2 \\ 0.6 & 1 \end{bmatrix}}{7} = \frac{7}{7} = 1,\quad x_2 = \frac{\begin{bmatrix} 3 & 6 \\ 2 & 0.5 \end{bmatrix}}{7} = \frac{-10.5}{7} = -1.5$$

10.8.2 Inverse of Matrix Method

Since AX = B, then X = A^{-1}B. That if we can find the inverse of matrix A in one of the methods mentioned in subsection (10.6), then the multiplication of A^{-1} with B give the unknowns columns X.

Example (1-6)

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -4/15 & 2/15 & -1/5 \\ 2/15 & -1/15 & -2/5 \\ -1/15 & -2/15 & -2/15 \end{bmatrix}$$

$$X = \begin{bmatrix} -4/15 & 2/15 & -1/5 \\ 2/15 & -1/15 & -2/5 \\ -1/15 & -2/15 & -2/15 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$x_1=1$, $x_2=2$ and $x_3=-1$.