1.5 Converting Volterra integral equation to ODE:

We shall present the technique that converts Volterra integral equations of second kind to equivalent ordinary differential equations.
This may be achieved by using the Leibnitz rule of differentiating the integral $\int_{a(x)}^{b(x)} F(x,t)\,dt$ with respect to x, we obtain

$$\frac{d}{dt}\int_{a(x)}^{b(x)} F(x,t)\,dt$$
$$= \int_{a(x)}^{b(x)} \frac{\partial F(x,t)}{\partial x}dt + \frac{db(x)}{dt}F\big(x,b(x)\big) - \frac{da(x)}{dt}F\big(x,a(x)\big)$$

where $F(x,t)$ and $\frac{\partial F}{\partial x}(x,t)$ are continuous functions of x and t in the domain $\alpha \le x \le \beta$and $t_0 \le x \le t_1$; and the limits of integration $a(x)$ and $b(x)$ are defined functions having continuous derivatives for $\alpha \le x \le \beta$.

A simple illustration is presented below:

$$\frac{d}{dt}\int_0^x sin(x-t)\,u(t)dt$$
$$= \int_0^x cos(x-t)u(t)\,dt + \left(\frac{dx}{dx}\right)\big(sin(x-x)u(x)\big)$$
$$- \left(\frac{d0}{dx}\right)\big(sin(x-0)u(0)\big)$$
$$= \int_0^x cos(x-t)u(t)\,dt.$$

Example_1: Reduce each of the Volterra integral equations to an equivalent initial value problem:

a) $u(x) = x - cos(x) + \int_0^x (x-t)\,u(t)\,dt$
b) $u(x) = x^4 + x^2 + 2\int_0^x (x-t)^2\,u(t)\,dt$
c) $u(x) = x^2 + \frac{1}{6}\int_0^x (x-t)^3\,u(t)\,dt$

Solution: (a) $u(x) = x - cos(x) + \int_0^x (x-t)\,u(t)\,dt$

$$\frac{d}{dx}\left[u(x) = x - cos(x) + \int_0^x (x-t)\,u(t)\,dt\right]$$

$$\boldsymbol{u}'(x) = 1 + sin(x) + \frac{d}{dx}\int_0^x (x-t)\,u(t)\,dt$$

$$\boldsymbol{u}'(x) = 1 + sin(x) + \int_0^x u(t)\,dt + \left(\frac{dx}{dx}\right)\,\big((x-x)u(x)\big) - \left(\frac{d0}{dx}\right)\,\big((x-0)u(0)\big)$$

$$\boldsymbol{u}'(x) = 1 + sin(x) + \int_0^x u(t)\,dt$$

$$\boldsymbol{u}''(x) = cos(x) + \int_0^x \frac{\partial u(t)}{\partial x} dt + \left(\frac{dx}{dx}\right)\,\big(u(x)\big) - \left(\frac{d0}{dx}\right)\,\big(u(0)\big)$$

$$\boldsymbol{u}''(x) = cos(x) + \ u(x)$$

$$\boldsymbol{u}''(x) - u(x) = cos(x)$$

$$\Longrightarrow u(0) = 0 - cos(0) + \int_0^0 (0-t)\,u(t)\,dt \ \Rightarrow u(0) = -1$$

$$\Longrightarrow \boldsymbol{u}'(0) = 1 + sin(0) + \int_0^0 u(t)\,dt \ \Longrightarrow \boldsymbol{u}'(0) = 1$$

$$\therefore\ \boldsymbol{u}''(x) - u(x) = cos(x) \quad with \quad u(0) = -1,\ \boldsymbol{u}'(0) = 1$$

1.6 Converting IVP to Volterra integral equation:

In this section we will study how an initial value problem (IVP) can be transformed to an equivalent Volterra integral equation.

Let us consider the integral equation

$$y(t)=\int_0^t f(t)\,dt \qquad (14)$$

The Laplace transform of $f(t)$ is defined as $\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}\,f(t)dt=F(s)$. Using this definition, equation (14) can be transformed to $\mathcal{L}\{y(t)\}=\frac{1}{s}\mathcal{L}\{f(t)\}$.

In a similar manner, if $y(t)=\int_0^t\int_0^t f(\tau)\,d\tau d\tau$, then $\mathcal{L}\{y(t)\}=\frac{1}{s^2}\mathcal{L}\{f(t)\}$.

This can be inverted by using the convolution theorem to yield

$$y(t)=\int_0^t (t-\tau)f(\tau)d\tau$$

If $y(t)=\underbrace{\int_0^t\int_0^t\dots\int_0^t f(\tau)\,d\tau\dots d\tau d\tau}_{n-fold\ integrals}$, then $\mathcal{L}\{y(t)\}=\frac{1}{s^n}\mathcal{L}\{f(t)\}$.

Using the convolution theorem, we get the Laplace inverse as $y(t)=\int_0^t\frac{(t-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau$

Thus the *n*-fold integrals can be expressed as a single integral in the following manner:

$$\int_0^t\int_0^t\dots\int_0^t f(\tau)\,d\tau\dots d\tau d\tau=\int_0^t\frac{(t-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau \qquad (15)$$

This is an essential and useful formula that has enormous applications in the integral equation problems.

Example_2: Derive an equivalent Volterra integral equations to each of the following initial value problems:

a) $\boldsymbol{y}''(x)+\boldsymbol{5y}'(x)+6y(x)=0 \quad with \quad y(0)=1,\ \boldsymbol{y}'(0)=1$
b) $\boldsymbol{y}''(x)+\boldsymbol{y}(x)=sin(x) \quad with \quad y(0)=0,\ \boldsymbol{y}'(0)=0$
c) $\boldsymbol{y}'''(x)+\boldsymbol{4y}'(x)=x \quad with \quad y(0)=0,\ \boldsymbol{y}'(0)=0,\boldsymbol{y}''(0)=1$

Solution: (a) $\boldsymbol{y}''(x)+\boldsymbol{5y}'(x)+6y(x)=0 \quad with \quad y(0)=1,\ \boldsymbol{y}'(0)=1$

Let $y''(x) = u(x)$ ... 1 $s.t.$ $u(x)$ $constant\ function$

By integrating both sides of **1** w.r.t. x from 0 to x

$\int_0^x y''(t)dt = \int_0^x u(t)dt \Rightarrow y'(x) - y'(0) = \int_0^x u(t)dt$

$\Rightarrow y'(x) = 1 + \int_0^x u(t)dt$ ... **2**

Similarly, by integrating both sides of **2** w.r.t. x from 0 to x

$\int_0^x y'(t)dt = \int_0^x [1 + \int_0^x u(t)dt]\ dt \Rightarrow y(x) - y(0) = x + \int_0^x \int_0^x u(t)dtdt$

$\Rightarrow y(x) = 1 + x + \int_0^x \int_0^x u(t)dtdt$ ... 3 using the convolution theorem

$\Rightarrow y(x) = 1 + x + \int_0^x (x - t)u(t)dt$ ... 4

Substituting **4, 2& 1** into $y''(x) + 5y'(x) + 6y(x) = 0$

$\Rightarrow u(x) + 5(1 + \int_0^x u(t)dt) + 6(1 + x + \int_0^x (x - t)u(t)dt) = 0$

$\Rightarrow u(x) = -11 - 6x - \int_0^x (5 + 6(x - t))u(t)dt$

Solution: (c)

$y'''(x) + 4y'(x) = x$ $with$ $y(0) = 0,\ y'(0) = 0, y''(0) = 1$

Let $y'''(x) = u(x)$ ... 1 $s.t.$ $u(x)$ $constant\ function$

By integrating both sides of **1** w.r.t. x from 0 to x

$\int_0^x y'''(t)dt = \int_0^x u(t)dt \Rightarrow y''(x) - y''(0) = \int_0^x u(t)dt$

$\Rightarrow y''(x) = 1 + \int_0^x u(t)dt$ ... **2**

Similarly, by integrating both sides of **2** w.r.t. x from 0 to x

$\int_0^x y''(t)dt = \int_0^x [1 + \int_0^x u(t)dt]\ dt \Rightarrow y'(x) - y'(0) = x + \int_0^x \int_0^x u(t)dtdt$

$\Rightarrow y'(x) = x + \int_0^x \int_0^x u(t)dt\,dt$ ... **3** using the convolution theorem

$\Rightarrow y'(x) = x + \int_0^x (x - t)u(t)dt$ ... 4

Substituting **1 & 4** into $y'''(x) + 4y'(x) = x$

$\Rightarrow u(x) + 4(x + \int_0^x (x - t)u(t)dt) = x$

$\Rightarrow u(x) = -3x - 4\int_0^x (x - t)u(t)dt$