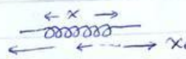
« CH-3 The Harmonic « Oscillator »

1- Harmonic Motion:-

For a linear case an example is a Force exerted on spring that's obyes Hooke's Law:

$F = -k\,(x - x_e)$ --- (1)



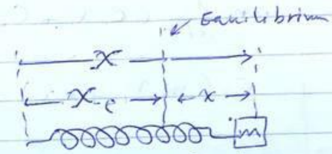
x = total length of spring

x_e = unstretched or equilibrium length

k = constant is called «stiffness»

$x - x_e$ = the displacement from equilibrium where ($F=0$) then

$F(x) = -k\,x$ --- (2)



fig(1)

If such a Force exerted on a body of mass m then Newton's 2nd Law is:

$m\ddot{x} = -kx$ or $m\ddot{x} + kx = 0$ --- (3)

to solve (equ 3) using the «trial method» where e^{qt} is the trial solution and q is constant.

$\therefore\ x = e^{qt}$ then:

sol

$m\dfrac{d^2}{dt^2}e^{qt} + k\,e^{qt} = 0$ by differentiating and canceling the common factor then

$mq^2 + k = 0 \Rightarrow q^2 = \dfrac{-k}{m}$ and

$q = \mp i\sqrt{\dfrac{k}{m}} = \mp i\omega_0$

the general ~~with order~~ $i = \sqrt{-1}$ $\rightarrow$ $\omega_0 = \sqrt{\dfrac{k}{m}}$ --- (4)

then the trial method yields two sol $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$

According to Euler theorem the complex exponential is:-

$$e^{iu} = \cos u + i \sin u$$
$$e^{-iu} = \cos u - i \sin u \quad \text{--- (5)}$$

or

$$\cos u = \frac{e^{iu} + e^{-iu}}{2} \quad , \quad \sin u = \frac{e^{iu} - e^{-iu}}{2i} \quad \text{--- (5-a)}$$

Solu ②

For Linear differential eqns. if $x_1(t)$ and $x_2(t)$ are known Solu then any Linear Combination $C_1 x_1 + C_2 x_2$ is also a Solus. C_1 and C_2 are constant. then:

$$m \frac{d^2}{dt^2}(C_1 x_1 + C_2 x_2) + k(C_1 x_1 + C_2 x_2) = 0$$

$$C_1(m\ddot{x}_1 + kx_1) + C_2(m\ddot{x}_2 + kx_2) = 0$$ since x_1 and x_2 are Solutions.

Solu ② 2-1 For general Solu:

$$x(t) = C_+ e^{i\omega_0 t} + C_- e^{-i\omega_0 t} \quad \text{--- (6)} \quad \underline{\text{or}}$$

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t \quad \text{--- (7)}$$ in which

$$C_+ = \frac{1}{2}(A - iB) \quad , \quad C_- = \frac{1}{2}(A + iB)$$

we want $x(t)$ and A and B to be real so C_+ and C_- are Conj Complex quantities.

Sol ③ A Third way of solution is found by use of the trigonometric identity $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ which yield

$$x(t) = \mathcal{A} \cos(\omega_0 t - \varphi) \quad \text{--- 8}$$ wher

$$A = \mathcal{A} \cos\varphi \quad \text{and} \quad B = \mathcal{A} \sin\varphi$$ so that

$$A^2 + B^2 = \mathcal{A}^2 \sin^2\varphi + \mathcal{A}^2 \cos^2\varphi = \mathcal{A}^2$$ and

$$\frac{B}{A} = \mathcal{A} \sin\varphi / \mathcal{A} \cos\varphi = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi$$

defenitions:

∴ The physical motion represented by our mathmatical solution is a sinusoidal oscillation of the displacement x as figu.
This is Known as «Harmonic motion»
The maximum displacement is called the «amplitude» of the oscilla
it is the constant A in (eq 8) or $(A^2+B^2)^{1/2}$ in (eqn 7)
w_0: is called the angular frequency
The «Period» T_0 of the oscillation: is the time riquired For one complete cycle, the time for which the argument of the sine or cosine terms increases by 2π. thus

$$T_0 w_0 = 2\pi \quad \text{or}$$

$$\boxed{T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}}} \qquad (9)$$

The «linear frequency»: the number of cycles per unit of time is denoted by f_0 and so

$$2\pi f_0 = w_0 \Rightarrow \boxed{f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}} \qquad (10)$$

ووحدة T_0 هي hertz او (الهرتز) Heinrich Hertz نسبة لمكتشفها

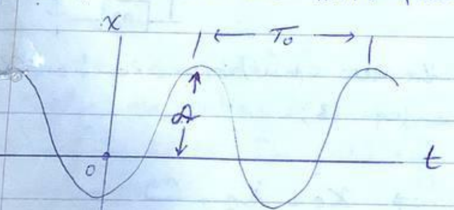


fig (2)

Constants of the motion and initial conditions:

The values of the constants in equs. (6, 7) are related to the initial conditions. Thus; if the block in fig(1) is initially displaced by an amount x_0 and its velocity is $\dot{x}_0$ at time $t=0$ then (eq 7) becomes: $x_0 = A$ → by differentiating (e.7)

$$\dot{x}(t) = -A w_0 \sin w_0 t + B w_0 \cos w_0 t$$

$$\therefore \dot{x}_0 = B w_0 \Rightarrow B = \frac{\dot{x}_0}{w_0}$$

The complete expression is:

$$x(t) = x_0 \cos \omega_0 t + \frac{\dot{x}_0}{\omega_0} \sin \omega_0 t \qquad (12)$$

The amplitude of the oscillation is:

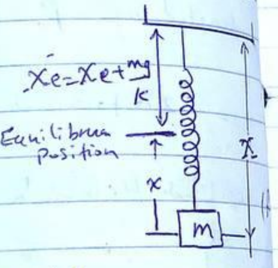
$$A = (A^2+B^2)^{\frac{1}{2}} = \left(x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2}\right)^{\frac{1}{2}}$$ then

$$C_+ = \frac{1}{2}\left(x_0 - i\frac{\dot{x}_0}{\omega_0}\right), \quad C_- = \frac{1}{2}\left(x_0 + i\frac{\dot{x}_0}{\omega_0}\right)$$

$\Phi = \tan^{-1}(\dot{x}_0 / x_0 \omega_0)$ the phase constant

Effect of a Constant External Force on A Harmonic oscillator

suppose the same spring is held in a vertical position as (fig. 3)



$$F = -k(X - X_e) + mg \quad (13)$$

$$x = X - \bar{X}_e$$

$$\therefore F = -kx + mg$$

x: is the displacement from the new equilibrium position $\bar{X}_e$ obtained by setting $F = 0$ in (equ. 13)

$$0 = -k(\bar{X}_e - X_e) + mg \Rightarrow \bar{X}_e = X_e + \frac{mg}{k}$$

$$\therefore x = X - \bar{X}_e = X - X_e - \frac{mg}{k}$$ Putting in (equ.

then

$F = -kx$ then its the same thing as the horizontal case

$$m\ddot{x} + kx = 0$$

then any constant external force applied to a harmonic oscillator merely shifts the equilibrium position.

Examples:
1- when a light
Position, the spring
unstretched len
distance D = f
t = 0, find (a)
when it passes
c) the accelerat
Sol
For the equili
$F_x = 0 = -$
$\therefore k =$
$\omega_0 = \sqrt{\frac{k}{m}}$
$x(t) = A\,c$
$\dot{x}(t) = -A$
we find
$x_0 =$
then the m
a) $x(t)$
$\therefore$ the ve
and the accele
As the block
argument of
(b) $\dot{x} = -$
c) (on - hal

Examples:

1- when a light spring supports a block of mass m in a vertical position, the spring is found to stretch by an amount D_1 over its unstretched length. If the block is further pulled down-ward a distance D_2 from the equilibrium position and released at time $t=0$, find (a) the resulting motion, (b) the velocity of the block when it passes back upward through the equilibrium position and c) the acceleration of the block at the top of its oscillatory motion

Sol

For the equilibrium position we have

$$F_x = 0 = -k D_1 + mg \qquad x \text{ (down ward)}$$

$$\therefore \ k = \frac{mg}{D_1}$$ then the angular frequency of ocilliation

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$ then the motion from $x(t)$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\dot{x}(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t$$ from the initial conditions we find

$$x_0 = D_2 = A \quad , \quad \dot{x}_0 = 0 = B\omega_0 \ , \quad B = 0$$

then the motion is:

a) $$x(t) = D_2 \cos\left(\sqrt{\frac{g}{D_1}}\, t\right)$$

$\therefore$ the velocity $$\dot{x}(t) = -D_2\sqrt{\frac{g}{D_1}}\,\sin\left(\sqrt{\frac{g}{D_1}}\, t\right)$$

and the acceleration $$\ddot{x}(t) = -D_2\frac{g}{D_1}\cos\left(\sqrt{\frac{g}{D_1}}\, t\right)$$

As the block passes upward through the equilibrium position the argument of the sine term is $\pi/2$ (one-quarter period) so

(b) $$\dot{x} = -D_2\sqrt{\frac{g}{D_1}} \ \text{(center)}$$, at the top of swing the argument of the cosine term (π)

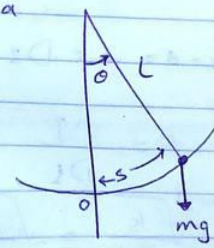
(c) (on-half period) then $$\ddot{x} = D_2\frac{g}{D_1} \ \text{(top)}$$ → يتبع

when $D_1 = D_2$ the downward acceleration at the top of the swing is just (g), this means that the block is in free fall that is the spring is exerting zero force on the block.

2. The Simple pendulum: it consists of a small "bob" of mass, swinging at the end of a light inextensible string of Length L as (fig.4) the motion is along a circular arc defined by the an[gle] θ. The restoring force is the component of the weight mg ac[ting] in the direction of increasing θ along the path of motion

$F_s = -mg\sin\theta$ treating to pop as a particle then

$m\ddot{s} = -mg\sin\theta$



(fig.4)

Now $s = L\theta$ and for small θ, $\sin\theta = \theta$

by sperating variable in terms of θ and L.

$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad \ddot{s} + \frac{g}{L}s = 0$

the motion is along a curved path rather than a straight lin[e] the differential eqns. is identical to that of the Linear harm[onic] ossillator, then (equ. 3) with the quantity g/L replacing k/m then the motion is simple harmonic with angular frequen[cy]

$\omega_0 = \sqrt{\frac{g}{L}}$ and Period $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}$

For $L = 1m \quad T_0 = 1$

when $T_0 = 2s \rightarrow L = \frac{g}{\pi^2}$ (Length of 2nd pendulum

$\therefore L = \frac{9.8062}{9.8696} = 0.9936\ m$

2- Energy Considerations in Harmonic motion:

Consider a particle under the action of a Linear restoring Force; $F = -kx$, Let us calculate the work done by an external Force F_{ext} in moving the particle from the equilibrium position $(x=0)$ to some position x. we have:

$$F_{ext} = -F_x = kx \quad \text{so}$$

$$W = \int_0^x F_{ext}\, dx = \int_0^x kx\, dx = \frac{1}{2} k x^2$$

In the case of a spring obeying Hooke's Law the work is stored in the spring as a potential energy: $W = V(x)$ then

$$V(x) = \frac{1}{2} k x^2 \quad \text{---} \quad (14)$$

$F_x = \frac{-dV}{dx} = -kx$, the total energy for a harmonic motion is:

$$E = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{kinetic energy}} + \frac{1}{2} \underbrace{k x^2}_{\text{potl. energy}} \quad \text{---} \quad (15)$$

اول A

the kinetic energy is quadratic in the velocity variable, and the potl. energy is quadratic in the displacement variable. The total energy is constant if there are no other forces except the restoring Force acting on the particle.

From (eq. 15) solving to finde the velocity:

$$\frac{dx}{dt} = \pm \left(\frac{2E}{m} - \frac{k}{m} x^2 \right)^{\frac{1}{2}} \quad \text{---} \quad (16)$$

$$dt = \int \frac{dx}{\sqrt{\frac{k}{m}} \left[\frac{2E}{k} - x^2 \right]^{\frac{1}{2}}} \quad \text{or} \quad t = \int \frac{dx}{\mp \left[(2E/m) - (\frac{k}{m}) x^2 \right]^{\frac{1}{2}}}$$

$$= \frac{dx}{\sqrt{\frac{k}{m}} \left[1 - (\frac{x}{A})^2 \right]^{\frac{1}{2}}} = \mp \left(\frac{m}{k} \right)^{\frac{1}{2}} \cos^{-1}\left(\frac{x}{A} \right) + C \quad \text{---} \quad 17$$

where : $A = \left(\frac{2E}{k}\right)^{\frac{1}{2}}$ —— (18)

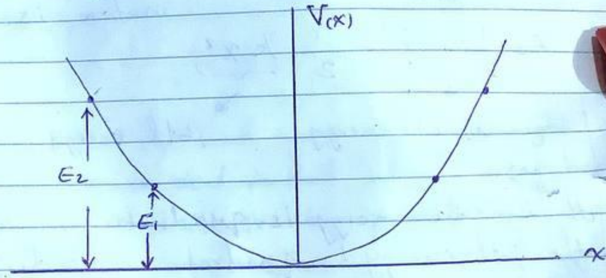
We can also find the amplitude from the (eqn. 15) by finding the turning points of the motion where ($\dot{x} = 0$):

the value of x must between $\left(\frac{2E}{k}\right)^{\frac{1}{2}}$ and $-\left(\frac{2E}{k}\right)^{\frac{1}{2}}$ in order for $\dot{x}$ to be real as (fig. 5)

The max. value of speed v_{max} occurs at $x = 0$ then:

$$E = \frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2 \quad —— (19)$$

As the particle oscillates, the kinetic and potl. energies continually change. the constant total energy as a form of kinetic energy at the center where $x = 0$ and $\dot{x} = \mp v_{max}$ and it is all potl. energy at the extrema where $\dot{x} = 0$ and $x = \mp A$.



(fig. 5)

Example:

③ : The energy function of the simple pendulum : the potl. energy of a simple pendulum is :

$$V = mgh$$

where h : is the vertical distance from the referenc level (level of the equilibrium position). For a displacment through an angle θ (fig. 4) we see that :

$$h = L - L \cos\theta \quad \text{then}$$

$$V(\theta) = mgL(1 - \cos\theta).$$

the series expansion for the cosines is:

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$, so for small θ we have:

$\cos\theta = 1 - \frac{\theta^2}{2}$ then

$V(\theta) = \frac{1}{2} mgL\,\theta^2$, $s = L\theta$ then

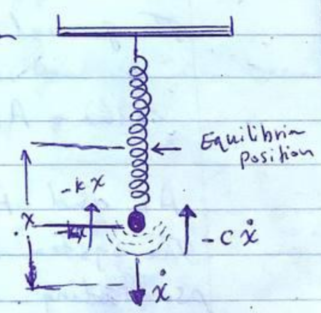
$V(s) = \frac{1}{2} mg \frac{s^2}{L}$ thus to first approximation the Potl. energy function is quadratic in the displacement variable. then the total energy is:

$$E = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} \frac{mg}{L} s^2$$

3. Damped Harmonic motion:

consider an object of mass m supported by a light spring of stiffness k. we shall assume that there is a viscous retarding force varying "linearly" with the velocity. as (fig. 6)

If x is the displacement from equilibrium then the restoring force is $-kx$ and the ratarding force $-c\dot{x}$ where c is constant of proportionallity, the diff. equ. of motion is:



(fig. 6)

$$m\ddot{x} = -kx - c\dot{x} \quad \text{or}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{——— (20)}$$

in the case of undamped force we use a trial solu. e^{qt} this is a solu. if:

$$m\frac{d^2}{dt^2} e^{qt} + c\frac{d}{dt} e^{qt} + k e^{qt} = 0 \quad \text{——— (21)}$$

For all t:

$mq^2 + cq + k = 0$ --- (22) the roots are given by the well-known formula:

$$q = \frac{-c \pm (c^2 - 4mk)^{\frac{1}{2}}}{2m} \quad \text{--- (23)} \quad (\text{الدستور})$$

depending on the value of the discriminant $(c^2 - 4mk)^{1/2}$:
there are three physically ~~d~~ cases:

I- $c^2 - 4mk > 0$ overdamping
II- $c^2 - 4mk = 0$ critical damping
III- $c^2 - 4mk < 0$ underdamping

I: for overdamped case, there are two different real (negative) values of q which we shall call $(-\gamma_1, -\gamma_2)$

$$q_1 = \frac{-c}{2m} + \left(\frac{c^2}{4m^2} - \frac{k}{m}\right)^{\frac{1}{2}} = -\gamma_1$$

$$q_2 = -\frac{c}{2m} - \left(\frac{c^2}{4m^2} - \frac{k}{m}\right)^{\frac{1}{2}} = -\gamma_2$$

The general solution is a linear combination:

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t} \quad \text{--- (24)}$$

A_1 and A_2 are determined from initial conditions:
The system returns to the equilibrium position without oscillating - the damping force dominates and prevents any periodic motion from taking place. as (fig-7)

II: In the case of critical damping the two roots are equal

$$q_1 = q_2 = \frac{-c}{2m} = -\gamma$$

Thus we have only on function, $e^{-\gamma t}$, then we need to find another soln. to satisfy the equ. of motion. to build up a general solution. $\Rightarrow$ then:

taking a boundary ...
we can back to ...
$= \gamma^2 m$, which ...

$m(\ddot{x} + 2\gamma\dot{x}$ ...

$(\frac{d}{dt} + \gamma)(\frac{d}{dt}$ ...

we now make ...

$(\frac{d}{dt} + \gamma) u =$ ...

$\therefore \frac{du}{u} = -$ ...

$u = A$ ...

$A e =$ ...

a 2nd in...
where B ...

$x(t) =$ ...

the motion ...

critical ...
an optim...
equilibri...
applica...
mechan...

taking a boundary conditions (initial displacement and velocity) we can back to (eq. 20) and re-write it by setting $\frac{c^2}{4m} = k = \gamma^2 m$, which is the condition for equal root of q. then:

$$m(\ddot{x} + 2\gamma\dot{x} + \gamma^2 x) = 0 \quad \text{then.}$$

$$\left(\frac{d}{dt} + \gamma\right)\left(\frac{d}{dt} + \gamma\right)x = 0$$

we now make the substitution : $u = \gamma x + \frac{dx}{dt}$ which gives:

$$\left(\frac{d}{dt} + \gamma\right)u = 0 \rightarrow \text{or} : \frac{du}{dt} = -\gamma u \qquad \text{by integrating:}$$

$$\therefore \frac{du}{u} = -\gamma\, dt \rightarrow \operatorname{Lin} u = -\gamma t + A' \quad \text{or}$$

$$u = A e^{-\gamma t}$$, A': is the constant of integration

$A = \operatorname{Lin} A'$, and from definition of u: then

$$A e^{-\gamma t} = \gamma x + \frac{dx}{dt} \quad \text{or:} \quad A = \left(\gamma x + \frac{dx}{dt}\right) e^{\gamma t} = \frac{d}{dt}(x e^{\gamma t})$$

a 2nd integration yields: $At + B = x e^{\gamma t}$

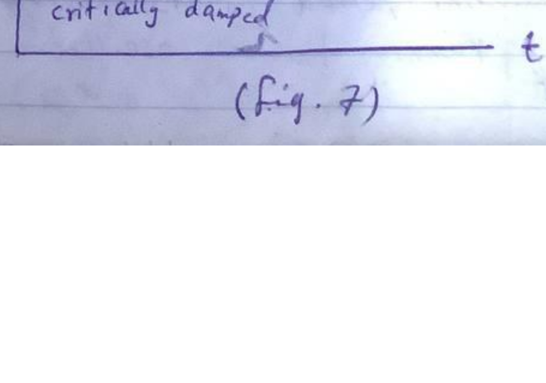
where B a 2nd constant of integration, the general Solu. becomes

$$x(t) = (At + B)e^{-\gamma t} = At e^{-\gamma t} + B e^{-\gamma t} \quad \text{———} (25)$$

the motion is non-oscillatory as in case I. (fig. 7).

critical damping gives an optimum return to equilibrium for certian applications, such as mechanical suspensions.

تعليق ↓ ثارة جل



(fig. 7)

III If the resistance constant c is small enough so that $c^2 < 4mk$ we have the case of underdamping
In this case: the quantity $c^2 - 4mk$ is (negative)
then the values of q are complex numbers.

$$q_1 = -\frac{c}{2m} + i\left(\frac{k}{m} - \frac{c^2}{4m^2}\right)^{\frac{1}{2}}$$

$$q_2 = -\frac{c}{2m} - i\left(\frac{k}{m} - \frac{c^2}{4m^2}\right)^{\frac{1}{2}}$$

Let us:

$$\omega_d = \left(\frac{k}{m} - \frac{c^2}{4m^2}\right)^{\frac{1}{2}} = \left(\omega_0^2 - \gamma^2\right)^{\frac{1}{2}} \quad \text{---} (26)$$

$$(\omega_d \lesssim \omega_0)$$

then:

$$q_1 = -\gamma + i\omega_d$$

where: $\gamma = \frac{c}{2m}$, $\omega_0 = \sqrt{\frac{k}{m}}$

$$q_2 = -\gamma - i\omega_d$$

the general solution of the equation of motion is a Linear combination:

$$x(t) = C_+ e^{(-\gamma + i\omega_d)t} + C_- e^{(-\gamma - i\omega_d)t} \quad \text{---} (27)$$

$$= e^{-\gamma t}\left(C_+ e^{i\omega_d t} + C_- e^{-i\omega_d t}\right)$$

(eq 27) as same of (equ. 6):
the general Soln:

$$x(t) = e^{-\gamma t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \quad \text{---} (28)$$

$$x(t) = e^{-\gamma t} \mathcal{A}\cos(\omega_d t - \phi) \quad \text{---} (29)$$

$$C_\mp = (A \mp iB)/2, \quad A^2 + B^2 = \mathcal{A}^2$$

and $\tan\phi = \frac{B}{A}$

the Period T_d is the natural Period of the Freely running harmonic oscillator with damping?

$$T_d = \frac{2\pi}{\omega_d} = 2\pi / (\omega_0^2 - \gamma^2)^{\frac{1}{2}} \quad \text{———} \quad (30)$$

a plot of the motion is shown in (fig. 8), (Equ. 29) shows that the two curves given by $x = A e^{-\gamma t}$ and $x = -A e^{-\gamma t}$ the cosine factor is between (+1) and (-1). at which points the curve of motion touches the envelope. The points of contact are separated by a time interval of one-half period $\frac{T_d}{2}$. In one complete period the amplitud diminishes by a factor $e^{-\gamma T_d}$

in a time $\gamma^{-1} = \frac{2m}{c}$ the amplitude decays by a factor $e^{-1} = 0.3679$. Then: In Summary: خلاصة

The Presence of damping of the linear type causes the oscillator given an initial motion to return to a state of rest at the equilibrium Position. The return is either oscillatory, or not, depending on the amount of damping. The critical condition, given by $c^2 = 4mk$, characterizes the limiting case of the non-oscillatory mode of ~~re~~ return.

(رسم) (fig. 8) يوجد رسم (p. 68)

Mechanical suspensions: For a mechanical system the type of damping is determined by the value of the mass. If the system is critically damped for a critical mass m_{crit}. We have $c^2 = 4k\,m_{crit}$. The discriminant is then

$$c^2 - 4km = 4k(m_{crit} - m) \quad \text{So:}$$

the system will be overdamped, or, underdamped depending on, whether the mass is Less than, or greater than m_{crit}.

Energy consideration: the total Energy for damped oscillator is:

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

This is Constant for the undamped oscillator, by differentiate E:

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x}$$

while: The differential equation of motion is:

$$m\ddot{x}+c\dot{x}+kx=0 \quad \text{or} \quad \text{Thus we can write}$$

$$m\ddot{x}+kx=-c\dot{x}$$

$$\frac{dE}{dt}=-c\dot{x}^2$$

its a product of the damping Force and the velocity. Its always either zero or negative then total energy continually decrease. The energy is dissipated as frictional heat by virtue of the viscous resistance to the motion.

Examples:

4- An automobile suspension system is critically damped, and its period of free oscillation with no damping is one second. If the system is initially displaced by an amount x_0, and released with zero initial velocity, find the displacement at $t=1s$.

solu:

For critical damping we have: $\gamma=\frac{c}{2m}=\left(\frac{k}{m}\right)^{\frac{1}{2}}=\omega_0=\frac{2\pi}{T_0}$

$\therefore \gamma=2\pi\ s^{-1}$ for $T_0=1s$. Now the general expression for the displacement: (eq.25) is:

$$x(t)=(At+B)e^{-\gamma t} \quad \text{So for } t=0$$

$$x(0)=B$$

differentiating: we have $\dot{x}(t)=(A-\gamma B-\gamma At)e^{-\gamma t}$

then $\dot{x}(0)=A-\gamma B=0$

So $A=\gamma B=\gamma x_0$ then

$$x(t)=x_0(1+\gamma t)e^{-\gamma t}=x_0(1+2\pi t)e^{-2\pi t}$$

is the displacement as a function of time.

For $t=1s \Rightarrow x(1)=(x_0+2\pi)e^{-2\pi}=x_0(7.28)e^{-6.28}=0.013x_0$

$\therefore$ the system has practically returned to equilibrium.

(5) - The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

Solu: we have $\omega_d = \frac{1}{2}\omega_0 = (\omega_0^2 - \gamma^2)^{\frac{1}{2}}$ which gives

$$\frac{\omega_0^2}{4} = \omega_0^2 - \gamma^2 \longrightarrow \gamma = \sqrt{\frac{3}{4}}\,\omega_0 = \omega_0\left(\frac{3}{4}\right)^{\frac{1}{2}}.$$

$\gamma T_d = \omega_0\left(\frac{3}{4}\right)^{\frac{1}{2}}\left[2\pi/(\omega_0/2)\right] = 10.88$ Thus the amplitude

ratio is: $e^{-\gamma T_d} = e^{-10.88} = 0.00002$ This is a highly damped oscillator

(6): Given: The terminal speed of a baseball in free fall is 30 m/s. Assuming a Linear air drag, Calculate the effect of air resistance on a simple Pendulum using a baseball as the "bob".

Solu In ch.2 we found the terminal speed for the case of Linear air drag which is $v_t = \frac{mg}{c_1}$ where c_1 is the Linear drag coefficient

$$\therefore\quad \gamma = \frac{c_1}{2m} = \frac{(mg/v_t)}{2m} = \frac{g}{2v_t} = \frac{9.8\,ms^{-2}}{60\,ms^{-1}} = 0.163\,s^{-1}$$

by a factor e^{-1} in a time $\gamma^{-1} = 6.13\,s$

from (ex.2) we have:

$$\omega_0 = \left(\frac{g}{L}\right)^{\frac{1}{2}} \text{ for small amplitude.}$$

then from (eq.30)

$$T_d = 2\pi(\omega_0^2 - \gamma^2)^{-\frac{1}{2}}$$
$$= 2\pi\left(\frac{g}{L} - 0.0265\right)^{-\frac{1}{2}}$$

For a basball « 2nd pendulum » for which the half-Period is one second in the absence of damping, we have $\frac{g}{L} = \pi^2$, then:

$$\frac{T_d}{2} = \pi(\pi^2 - 0.0265)^{-\frac{1}{2}} = 1.00134\ S.$$

4- Forced Harmonic Motion. Resonance:

We shall study the motion of a damped harmonic osillator that is driven by an external Force Fext. the total applied Force is then the sum of three Forces: the elastic restoring Force $-kx$, the viscous damping Force $-c\dot{x}$, and the driving Force Fext. then: the differential equation of motion is:

$$-kx - c\dot{x} + F_{ext} = m\ddot{x} \quad \text{or:} \quad m\ddot{x} + c\dot{x} + kx = F_{ext} \qquad (31)$$

when Fext varies sinusoidally with time then $F_{ext} = F_0 \cos\omega t$
where F_0: amplitude , ω: angular Frequency
its straightforward. and to solve (31) using the above trigonometric form for the driving Force:

$$\therefore \quad F_{ext} = F_0 e^{i\omega t} \qquad \text{so that}$$

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \qquad (32)$$

mathmatically the variable x is now a complex number. the solution of the above linear diff. eqn. is given by the sum of two parts: ⊛ the first is the solution of the homogeneous equation $m\ddot{x} + c\dot{x} + kx = 0$ (متجانسة)
⊛ the second is any particular solu. that we are able to find

<u>define</u>

the solution of the homogeneous equation represents a motion oscillatory or non oscillatory, which decays to zero - it called the "transient term". طفلي او مؤقت

For the steady-state condition, we shall tray a solution of the complex exponential type:

$$x(t) = A e^{i(\omega t - \phi)} \qquad (33)$$

where the amplitude A and the phase difference ϕ are constants if this "guess" is correct, we must have:

$$m\frac{d^2}{dt^2}Ae^{i(\omega t-\phi)}+c\frac{d}{dt}Ae^{i(\omega t-\phi)}+kAe^{i(\omega t-\phi)}=F_0e^{i\omega t}$$

by cancelling the factor $e^{i\omega t}$ we find:

$$-m\omega^2A+i\omega cA+kA=F_0e^{i\phi}=F_0(\cos\phi+i\sin\phi) \quad --(34)$$

Equating the real and imaginary parts yields the two equations:

$$\left.\begin{aligned} A(k-m\omega^2) &= F_0\cos\phi \quad --1 \\ c\omega A &= F_0\sin\phi \quad --2 \end{aligned}\right\} \quad --(35)$$

dividing (2) by (1) and using $\tan\phi=\sin\phi/\cos\phi$ then:

$$\tan\phi=\frac{c\omega}{k-m\omega^2} \quad --(36)$$

by squaring both sides of (equ. 35) and adding with using $\sin^2\phi+\cos^2\phi=1$ then.

$$A^2(k-m\omega^2)^2+c^2\omega^2A^2=F_0^2$$

solving for A, the amplitude of the study-state oscillation, as a function of the driving frequency:

$$A(\omega)=\frac{F_0}{[(k-m\omega^2)^2+c^2\omega^2]^{\frac{1}{2}}} \quad --(37)$$

we know that $\omega_0^2=\frac{k}{m}$, $\gamma=\frac{c}{2m}$ then (36)(37) becomes

$$\tan\phi=\frac{2\gamma\omega}{\omega_0^2-\omega^2} \quad --(38)$$

$$A(\omega)=\frac{F_0/m}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{\frac{1}{2}}} \quad --(39)$$

$$A(\omega)=\frac{F_0/m}{D(\omega)} \quad --(39\text{-}a)$$ where $D(\omega)=[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{\frac{1}{2}}$

The above result relating the amplitude and phase of the damped harmonic oscillator under the action of a sinusoidal driving Force.

a plot of $A(\omega)$ versus ω as (fig 9) shows that the amplitude assumes a maximum value at a certain applied frequency ω called the "amplitude resonant frequency".

The function $D(\omega)$ defined above is known as the "resonant denominator"

To find ω_r we calculate $dA/d\omega$ from equ (39) and set the result equal to zero, then finally obtain:

$$\omega_r = (\omega_0^2 - 2\gamma^2)^{\frac{1}{2}} \quad \text{---(40)} \quad \text{(home.work)}$$

In the case of weak damping, if γ is very small compared to ω_0, then the resonant frequency differs by only a small amount from ω_0, the frequency of the freely running undamped oscillator

The angular frequency of the freely running oscillator is given by:

$$\omega_d = (\omega_0^2 - \gamma^2)^{\frac{1}{2}} \quad \text{---(41)} \quad \overset{\text{same}}{\text{---(26)}}$$

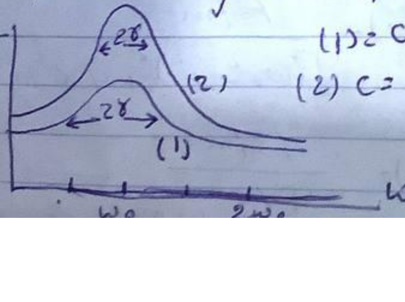
At the extreme of strong damping, no amplitude resonance occurs if $\gamma \geq \omega_0/\sqrt{2}$ because the amplitude then becomes a monotonically decreasing function of ω. To see this consider the limitting case $\gamma^2 = \frac{\omega_0^2}{2}$

(equ. 39) becomes:

$$A(\omega) = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + 2\omega_0^2\omega^2]^{\frac{1}{2}}} = \frac{F_0/m}{(\omega_0^4 + \omega^4)^{\frac{1}{2}}}$$

which clearly decreases with increasing values of ω starting with $\omega = 0$

Graphs showing amplitude versus driving freq. for two values of the damping constant.



(1): $C = 1/2\, m\omega_0$

(2) $C = 1/4\, m\omega_0$

(fig. 9)

Amplitude of oscillations at the resonance Peak:

We shall call the stady-state amplitude at the resonance freqency A_{max} and is obtained from eq. 39 and eq. 40. The result is:

$$A_{max} = \frac{F_o/m}{2\gamma\sqrt{\omega_o^2-\gamma^2}} = \frac{F_o}{c\,\omega_d} \quad \text{——} (42)$$

In the case of weak damping we can neglect γ^2 and write:

$$A_{max} \simeq \frac{F_o}{2\gamma m\omega_o} = \frac{F_o}{c\,\omega_o} \quad \text{——} (42a)$$

Thus the amplitude of the induced oscillations at the resonant condition becomes very large if the damping constant (c) is very small and conversely. In mechanical systems it may, or may ~~be~~ not, be desirable to have large resonant amplitudes. In the case of electric motors rubber or spring mounts are used to minimize the transmission of vibration. The stiffness of these mounts (مساند) is chosen so as to ensure that the resulting resonant frequency is far from the running frequency of the motor.

(حدة) Sharpness of the Resonance. Quality factor:

Let us consider the case of weak damping $\gamma << \omega_o$. Then in the expression for stady-state amplitude (equ. 39) we can make the following substitutions:

$$\omega_o^2-\omega^2 = (\omega_o+\omega)(\omega_o-\omega) \simeq 2\omega_o(\omega_o-\omega)$$

$$\gamma\omega \simeq \gamma\omega_o$$

Then:

$$A(\omega) \simeq \frac{A_{max}\,\gamma}{\sqrt{(\omega_o-\omega)^2+\gamma^2}}$$

$\therefore$ when:

$$|\omega_o-\omega| = \gamma \quad \text{if} \quad \omega = \omega_o \mp \gamma$$

then

$$A^2 = \frac{1}{2}A_{max}^2$$ this means that γ is measure of

the width of the resonance curve. Thus (2γ) is the frequency difference between the points for which the energy is down by a factor of $\frac{1}{2}$ from the energy at resonance, because the energy is proportional to A^2 af (fig. 9)

Another way of designating the sharpness of the resonance peak is in terms of a parameter Q called the "Quality factor" of a resonant system. It is defined as:

$$Q = \frac{\omega_d}{2\gamma} \quad (44), \text{ or for weak damping } Q \simeq \frac{\omega_0}{2\gamma}$$

thus the total width $\Delta\omega$ at the half-energy points is approximately

$$\Delta\omega = 2\gamma = \frac{\omega_0}{Q} \quad \text{or since } \omega = 2\pi f \text{ then}$$

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} = \frac{1}{Q} \quad (45)$$

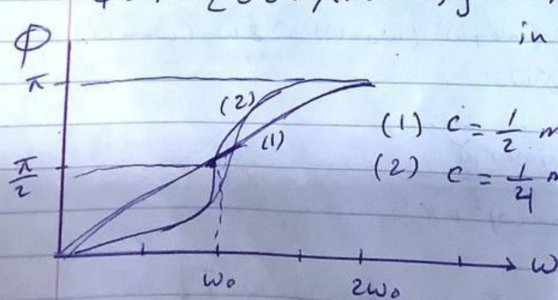
giving the fractional width of the resonance peak. In (fig. 9) the value of Q is 2 for curve (1) and 4 for curve (2).

Phase Angle:

The phase difference between the applied driving force and the steady-state response is given by (equ 38)

$$\phi = \tan^{-1}[2\gamma\omega/(\omega_0^2 - \omega^2)]$$

this relation is plotted in (fig. 10)



(1) $c = \frac{1}{2} m\omega_0$

(2) $c = \frac{1}{4} m\omega_0$

(fig. 10) Phase angle versus driving frequency for two values of the damping constant.

At $\omega = 0 \Rightarrow \phi =$ zero and remains small for small ω so the response is in phase with the driving Force.
Near the resonance frequency, at $\omega = \omega_0$, the phase angle ϕ increase to $\frac{\pi}{2}$ and the response is 90°.
For large values of ω the value of ϕ approaches π then the motion of the system is 180° out of phase with the driving Force.

Velocity Resonance:

The actual motion of the drivin harmonic oscillator is given by taking the real part of both sides of (equ. 33)

$$x(t) = A(\omega) \cos(\omega t - \phi) \quad \text{---} \quad (46)$$

where $A(\omega)$ given by (equs. 37, 39)
by differentiating (equ. 46) find:

$$\dot{x}(t) = -\omega A(\omega) \sin(\omega t - \phi) \quad \text{---} \quad (47)$$

$\omega A(\omega) \Rightarrow$ "velocity amplitude" of the driven harmonic oscillator

$\dot{x}(t)$ varies between $-\omega A(\omega)$ and $+\omega A(\omega) \xrightarrow{\text{then}} v(\omega)$

$$\therefore v(\omega) = \frac{\omega F_0/m}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{\frac{1}{2}}} \quad \text{---} \quad (48)$$

maximum value of $v(\omega)$ occurs at $\omega = \omega_0$ then ω_0 is the "velocity resonant frequency".
ω is the "amplitude resonant frequency" differs from ω_0 by only small amount in the case of weak damping.
But for strong damping we found that no amplitude resonance occurs when $\gamma > \omega_0/\sqrt{2}$
$v(\omega) = \omega A(\omega)$ exhibits a maximum at $\omega = \omega_0$ for <u>any</u> value of damping factor γ.

Electrical-Mechanical Analogs:

When an electric current flows in a circuit comprised of inductive, capacitative and resistive elements, there is a precise analog with a moving mechanical system of masses and springs with frictional forces of the type studied.

If a current $i = \frac{dq}{dt}$ flows through an inductance L, the potential difference across the inductance is $L\ddot{q}$ and the stored energy is $\frac{1}{2}L\dot{q}^2$. (inductance and charge are analogous to mass and displacement and potential difference is analogous to force).

If a capacitance C carries a charge q the potential difference is $(C^{-1}q)$ and the stored energy is $(\frac{1}{2}C^{-1}q^2)$ we see that $\frac{1}{C}$ is analogous to stiffness constant of spring.

For an electric current i flowing through resistance R, the potl. difference is $(iR = \dot{q}R)$ and the rate of energy dissipation is $(i^2R = \dot{q}^2R)$ is analogy with the quantity (c) for a mechanical system. as the table:

Mechanical		Electrical	
x	displacement	q	charge
$\dot{x}$	velocity	$\dot{q} = i$	current
m	mass	L	inductance
k	stiffness	C^{-1}	reciprocal of capacitance
c	damping resistance	R	Ressitance
F	Force	V	Potential difference

Example:

7) The exponential damping factor γ of a spring suspension is one-tenth the critical value. If the undamped frequency is ω_0. Find (a) the resonant frequency. (b) the quality factor. (c) the phase angle ϕ when the system is driven at a frequency $\omega = \frac{\omega_0}{2}$ and (d) the steady-state amplitude at this frequency?

solution: we have $\gamma = \gamma_{crit}/10 = \frac{\omega_0}{10}$ so from (equ. 40)

(a) $\omega_r = [\omega_0^2 - 2(\omega_0/10)^2]^{\frac{1}{2}} = \omega_0(0.98)^{\frac{1}{2}} = 0.99\,\omega_0$

(b) The system can be regarded as weakly damped. So from (eqn. 44a)

$$Q \simeq \frac{\omega_0}{2\gamma} = \frac{\omega_0}{2(\omega_0/10)} = 5$$

(c) From (equ. 38) we have $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right) = \cancel{\tan^{-1}}$

$$= \tan^{-1}\left[\frac{2(\omega_0/10)(\omega_0/2)}{\omega_0^2-(\omega_0/2)^2}\right] = \tan^{-1} 0.133 = 7.6^\circ$$

d) From (equ. 39) and (39.a) we first calculate the value of the resonance denominator

$$D(\omega = \omega_0/2) = [(\omega_0^2 - \omega_0^2/4)^2 + 4(\omega_0/10)^2(\omega_0/2)^2]^{\frac{1}{2}}$$
$$= [(9/16) + (1/100)]^{\frac{1}{2}}\omega_0^2 = 0.7506\,\omega_0^2$$

From this the amplitude is:

$$A(\omega = \omega_0/2) = \frac{F_0/m}{0.7506\,\omega_0^2} = 1.332\,\frac{F_0}{m\omega_0^2}$$

Notice that the Factor $(F_0/m\omega_0^2) = F_0/k$ is steady-state amplitude for zero driving freqency.

~~4~~. The Nonlinear oscillator. Method of successive Approximations:

5: NonSinusoidal Driving Force: Fourier series:

In order to determine the motion of a harmonic oscillator that is driven by an external Periodic force which is "other than "Pure" Sinusoidal, it is necessary to know more methods than tha in section 4. We will use the principle of "SuperPosition".

If the external driving Force acting on a damped harmonic oscillator is given by a superposition of force functions

$$F_{ext} = \sum_n F_n(t) \quad \text{---} \quad (49)$$

such that the differential equations:

$$m\ddot{x}_n + c\dot{x}_n + k x_n = F_n(t) \quad \text{then the Solutions}$$

$$m\ddot{x} + c\dot{x} + kx = F_{ext} \text{ ——— (50) is given}$$

by the superposition: $x(t) = \sum_n x_n(t)$ then —(51)

$$m\ddot{x} + c\dot{x} + kx = \sum_n (m\ddot{x}_n + c\dot{x}_n + kx) = \sum_n F_n(t)$$

$$= F_{ext}$$

when the driving force is periodic, if for any value of time t:

$$F_{ext}(t) = F_{ext}(t + T)$$

where T is the period, then the force function can be expressed as a superposition of harmonic term according to Fourier's theorem which state that any periodic function $f(t)$ can be expanded as a sum as follows:

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right] \text{ —(52)}$$

where the coefficients:

$$\left.\begin{aligned} a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t)\, dt \qquad n = 0, 1, 2, \dots \\ b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t)\, dt \qquad n = 1, 2, \dots \end{aligned}\right\} \text{ —(53)}$$

where: T is period and $\omega = \frac{2\pi}{T}$ is the fundamental frequency

If $f(t)$ is an even function $\Rightarrow f(t) = f(-t)$ then

$b_n = 0$ for all n. The series expansion is known as a « Fourier Cosine Series ».

If $f(t)$ an odd function $\Longrightarrow f(t) = -f(-t)$ then all a_n vanish and the series is called "Fourier Sine series".

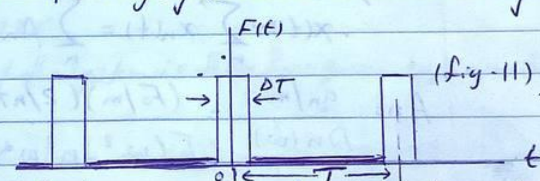
By use $e^{iu} = \cos u + i \sin u$, then equs. (52, 53) may also be expressed in complex exponential form as follows:

$$f(t) = \sum_n c_n e^{in\omega t} \qquad n = 0, \mp 1, \mp 2, \cdots \quad (54)$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-in\omega t}\, dt \quad (55)$$

Thus to find the steady-state motion of our harmonic oscillatory we express the force as a Fourier series of the form of equs (52, 54) using (53, 55) to determine the coefficients a_n and b_n, c_n. For each value of n and frequency ω, there is a response function $x_n(t)$. This function is the steady state solution of the driven oscillator. The superposition of all the $x_n(t)$ gives the actual motion. As a result, if the damping constant γ is very small, the resulting oscillation may be very nearly sinusoidal even if a highly nonsinusoidal driving force is applied.

Example:



(fig-11)

(8) Periodic Pulse: The motion of a harmonic oscillator that is driven by an external force consisting of a succession of rectangular pulses:

$$F_{ext} = F_0 \qquad NT - \frac{1}{2}\Delta T \leq t \leq NT + \frac{1}{2}\Delta T$$

$$F_{ext} = 0 \qquad \text{otherwise}$$

where $N = 0, \mp 1, \mp 2, \cdots$, T, is the time from one pulse to the next ΔT is the width of each pulse as shown in fig 11). In this case $F_{ext}(t)$ is an even function of $f(t)$ so it can be express as a fourier cosine series then eqn. 53 gives:

$$a_n=\frac{2}{T}\int_{-\Delta T/2}^{\Delta T/2} F_0\cos(n\omega t)\,dt=\frac{2}{T}F_0\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\Delta T/2}^{\Delta T/2}$$

$$=F_0\,\frac{2\sin\left(n\pi\frac{\Delta T}{T}\right)}{n\pi} \quad \text{where n is last step we use that } \omega=2\pi/T \qquad (56)$$

then: $a_0=\frac{2}{T}\int_{-\Delta T/2}^{\Delta T/2}F_0\,dt=F_0\,\frac{2\Delta T}{T}$

thus for our periodic pulse force we can write:

$$F_{ext}(t)=F_0\left[\frac{\Delta T}{T}+\frac{2}{\pi}\sin\left(\pi\frac{\Delta T}{T}\right)\cos(\omega t)+\frac{2}{2\pi}\sin\left(2\pi\frac{\Delta T}{T}\right)\cos(2\omega t)+\frac{2}{3\pi}\sin\left(3\pi\frac{\Delta T}{T}\right)\cos(3\omega t)+\dots\right] \qquad (57)$$

The first term is just the "average" value of the external force $F_{ave}=F_0\left(\frac{\Delta T}{T}\right)$. The second term is the fourier component at frequency ω. The remaining terms are harmonics of the fundamental: $2\omega, 3\omega$, and so on.

From eqns. (38, 39) we can write the final expression of the motion of our pulse-driven oscillator by use super position principle:

$$x(t)=\sum_n x_n(t)=\sum_n A_n\cos(n\omega t-\phi_n) \qquad (58)$$

$$A_n=\frac{a_n/m}{D_n(\omega)}=\frac{(F_0/m)(2/n\pi)\sin(n\pi\Delta T/T)}{\left[(\omega_0^2-n^2\omega^2)^2+4\gamma^2n^2\omega^2\right]^{\frac{1}{2}}} \qquad (59)$$

and the phase angles:

$$\phi_n=\tan^{-1}\left(\frac{2\gamma n\omega}{\omega_0^2-n^2\omega^2}\right) \qquad (60)$$

where m is the mass, γ is the decay constant, ω_0 is frequency of the freely running oscillator with no damping.

مثال Let us consider the spring suspension system of (example 7) under the action of a periodic pulse for which the pulse width is one-tenth the pulse period: $\frac{\Delta T}{T}=0.1$ and $\gamma=0.1\,\omega_0$ and $\omega=\omega_0/2$

The ~~Fast~~ Fourier series as equ. 57 becomes:

$$F_{ext}(t) = F_0\Big[0.1 + \frac{2}{\pi}\sin(0.1\pi)\cos(\omega t) + \frac{2}{2\pi}\sin(0.2\pi)\cos(2\omega t) + \frac{2}{3\pi}\sin(0.3\pi)\cos(3\omega t) + \cdots\Big]$$

$$= F_0\big[0.1 + 0.197\cos(\omega t) + 0.187\cos(2\omega t) + 0.172\cos(3\omega t) + \cdots\big]$$

The resonance denominators in equ (59) are given by:

$$D_n = \Big[\Big(\omega_0^2 - n^2\frac{\omega_0^2}{4}\Big)^2 + 4(0.1)^2\omega_0^2 n^2\frac{\omega_0^2}{4}\Big]^{\frac{1}{2}} = \Big[\Big(1-\frac{n^2}{4}\Big)^2 + 0.01n^2\Big]^{\frac{1}{2}}\omega_0^2$$

Thus $D_0 = \omega_0^2$ $D_1 = 0.757\omega_0^2$ $D_2 = 0.2\omega_0^2$ $D_3 = 1.285\omega_0^2$

the phase angles:

$$\phi_n = \tan^{-1}\Big(\frac{0.2n\omega_0^2/2}{\omega_0^2 - n^2\omega_0^2/4}\Big) = \tan^{-1}\Big(\frac{0.4n}{4-n^2}\Big)$$

which gives:

$$\phi = 0 \qquad \phi_1 = \tan^{-1}(0.1) = 0.0997 \qquad \phi_2 = \tan^{-1}\infty = \pi/2$$

$$\phi_3 = \tan^{-1}(-0.24) = -0.226$$

The steady state motion of the system is therefore given by (equ. 58)

$$x(t) = \frac{F_0}{m\omega_0^2}\big[0.1 + 0.26\cos(\omega t - 0.997) + 0.935\sin(2\omega t) + 0.134\cos(3\omega t + 0.226) + \cdots\big]$$

The dominant term is the one involving the second harmonic $2\omega = \omega_0$ because ω_0 is close to resonant frequency. Note also the phase of this term: $\cos(2\omega t - \pi/2) = \sin(2\omega t)$.

Problems of ch. 3 (20)