Lecture (3)

Finite Difference Methods (Part 1)

The finite difference method (FDM) is an approximate method for solving partial differential equations. It has been used to solve a wide range of problems including linear and non-linear problems. This method can be applied to problems with different boundary shapes and different boundary conditions.

3.1 One dimension finite difference method

Consider the derivative $\frac{df}{dx}$, where $f = f(x)$, and x is the independent variable (which could be either space or time). Finite-difference methods represent the continuous function $f(x)$ by a set of values defined at a finite number of discrete points in a specified (spatial or temporal) region. Thus, we usually introduce a "grid" with discrete points where the variable f is defined,
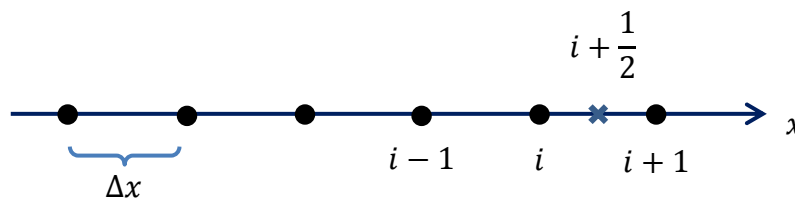

Figure 3.1 An example of a grid, with uniform spacing Δx (grid size).

We assume that the grid spacing is constant, then $x_i = i\Delta x$, where i is the index used to identify the grid points. Using the notation $f_i = f(x_i) = f(i\Delta x)$, we can define

The *forward difference* at the point i by $f_{i+1} - f_i$,   (3.1)

The *backward difference* at the point i by $f_i - f_{i-1}$,   (3.2)

The *centered difference* quotient at the point $i + \frac{1}{2}$ by $f_{i+1} - f_i$,   (3.3)

Note that f itself is not defined at the point $i + \frac{1}{2}$. From (3.1) - (3.3) we can define the following "finite-difference quotients:"

The forward-difference quotient at the point i :

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_{i+1} - f_i}{\Delta x} \; ; \qquad (3.4)$$

The backward – difference quotient at the point i:

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_i - f_{i-1}}{\Delta x} \; ; \qquad (3.5)$$

and the centered-difference quotient at the point $i + \frac{1}{2}$:

$$\left(\frac{df}{dx}\right)_{i+\frac{1}{2},\text{approx}} = \frac{f_{i+1} - f_i}{\Delta x} \; ; \qquad (3.6)$$

In addition, the centered-difference quotient at the point i can be defined by

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} \; ; \qquad (3.7)$$

Since (3.4) and (3.5) employ the values of f at two points, and give an approximation to $\frac{df}{dx}$ at one of the same points, they are sometimes called *two-point approximations*. On the other hand, (3.6) and (3.7) are *three-point approximations*, because the approximation to $\frac{df}{dx}$ is defined at a location different from the locations of the two values of f on the right-hand side of the equals sign.

How accurate are these finite-difference approximations?

"Accuracy" can be measured in a variety of ways, as we shall see. One measure of accuracy is *truncation error*. The term refers to the truncation of an infinite series expansion. As an example, consider the forward difference quotient

$$\left(\frac{df}{dx}\right)_{i,\text{approx}} = \frac{f_{i+1} - f_i}{\Delta x} \equiv \frac{f[(i+1)\Delta x] - f(i\Delta x)}{\Delta x} \; ; \qquad (3.8)$$

Expand f in a Taylor series about the point x_i , as follows:

$$f_{i+1} = f_i + \Delta x \left(\frac{df}{dx}\right)_i + \frac{(\Delta x)^2}{2!}\left(\frac{d^2 f}{dx^2}\right)_i + \frac{(\Delta x)^3}{3!}\left(\frac{d^3 f}{dx^3}\right)_i + \cdots + \frac{(\Delta x)^{n-1}}{(n-1)!}\left(\frac{d^{n-1} f}{dx^{n-1}}\right)_i + \cdots \quad (3.9)$$

Eq. (3.9) can be rearranged to

$$\frac{f_{i+1} - f_i}{\Delta x} = \left(\frac{df}{dx}\right)_i + \varepsilon \qquad (3.10)$$

where

$$\varepsilon \equiv \frac{(\Delta x)^2}{2!}\left(\frac{d^2 f}{dx^2}\right)_i + \frac{(\Delta x)^3}{3!}\left(\frac{d^3 f}{dx^3}\right)_i + \cdots + \frac{(\Delta x)^{n-1}}{(n-1)!}\left(\frac{d^{n-1} f}{dx^{n-1}}\right)_i + \cdots \quad (3.11)$$

is called the *truncation error*.

If Δx is small enough, the leading term on the right-hand side of Eq (3.11) will be the largest part of the error. *The lowest power of Δx that appears in the truncation error is called the "order of accuracy" or "order of approximation" of the corresponding difference quotient.* For example, the leading term of (3.10) is of order Δx, abbreviated as $O(\Delta x)$, and so we say that (3.10) is a first order approximation or an approximation of first-order accuracy. Obviously (3.5) is also first-order accurate. Just to be as clear as possible, a first-order scheme for the first derivative has the form $\left(\frac{df}{dx}\right)_{i,\text{approx}} = \left(\frac{df}{dx}\right)_i + O[\Delta x]$, where $\left(\frac{df}{dx}\right)_{i,\text{approx}}$ is an *approximation* to the first derivative and $\left(\frac{df}{dx}\right)_i$ is the true first derivative. Similarly, a second- order scheme for the first derivative has the form $\left(\frac{df}{dx}\right)_{i,\text{approx}} = \left(\frac{df}{dx}\right)_i +$ $O[(\Delta x)^2]$, and so on for higher orders of accuracy.

Similar analyses of (3.6) and (3.7) show that they are of second-order accuracy. For example, we can write

$$f_{i-1} = f_i + \left(\frac{df}{dx}\right)_i (-\Delta x) + \left(\frac{d^2 f}{dx^2}\right)_i \frac{(-\Delta x)^2}{2!} + \left(\frac{d^3 f}{dx^3}\right)_i \left[\frac{-(\Delta x)^3}{3!}\right] + \cdots \qquad (3.12)$$

Subtracting (3.12) from (3.9) gives

$$f_{i+1} - f_{i-1} = 2\left(\frac{df}{dx}\right)_i (\Delta x) + \frac{2}{3!}\left(\frac{d^3 f}{dx^3}\right)_i [(\Delta x)^3] + \cdots \quad \text{odd powers only,} \qquad (3.13)$$

which can be rearranged to

$$\left(\frac{df}{dx}\right)_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \left(\frac{d^3 f}{dx^3}\right)_i \frac{\Delta x^2}{3!} + O[(\Delta x)^4] \qquad (3.14)$$

Similarly,

$$\left(\frac{df}{dx}\right)_{i+\frac{1}{2}} \cong \frac{f_{i+1} - f_i}{\Delta x} - \left(\frac{d^3 f}{dx^3}\right)_{i+\frac{1}{2}} \frac{(\Delta x/2)^2}{3!} + O[(\Delta x)^4]. \qquad (3.15)$$

From (3.14) and (3.15), we see that

$$\left|\frac{Error\ of\ (3.14)}{Error\ of\ (3.15)}\right| \cong \frac{\left(\frac{d^3f}{dx^3}\right)_i \frac{\Delta x^2}{3!}}{\left(\frac{d^3f}{dx^3}\right)_{i+\frac{1}{2}} \frac{(\Delta x/2)^2}{3!}} = \frac{4\left(\frac{d^3f}{dx^3}\right)_i}{\left(\frac{d^3f}{dx^3}\right)_{i+\frac{1}{2}}} \cong 4 \qquad (3.16)$$

This shows that the error of (3.14) is about four times as large as the error of (3.15), even though both finite-difference quotients have second-order accuracy. The point is that the *"order of accuracy" tells how rapidly the error changes as the grid is refined, but it does not tell how large the error is for a given grid size.* It is possible for a scheme of low-order accuracy to give a more accurate result than a scheme of higher-order accuracy, if a finer grid spacing is used with the low-order scheme.

Exercises

Q1. Define: Finite difference Method, Truncation error, Order of accuracy.

Q2. (IMP.) Prove that the error in centered-difference quotient at i is four times the error in the centered-difference quotient at the point $+\frac{1}{2}$.

MATLAB Work

Write Matlab script to calculate the air pressure for the following temperatures (20, 15, 10, 5, 0, -5, -10, -20, -40, -60) ºC by using iteration loop, then draw the results. (Note: Use ideal gas law in Lecture 2).

Homework

1. The centered-difference quotient (Eq 3.7) is more accurate than the forward- and backward-difference quotients (Eq 3.4 and 3.5, respectively). Prove that.

2. Are the equations (3.4-3.16) ordinary differential equations or partial differential equations and why?