Lesson 1 – Math Review

Partial Derivatives and Differentials

- The differential of a function of two variables, $f(x, y)$, is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{1}$$

  - Eq. (1) is true regardless of whether *x* and *y* are independent, or if they are both composite functions depending on a third variable, such as *t*.
- The terms like $\partial f / \partial x$ and $\partial f / \partial y$ are called partial derivatives, because they are taken assuming that all other variables besides that in the denominator are constant.
  - For example, $\partial f / \partial x$ describes how *f* changes as *x* changes (holding *y* constant), and $\partial f / \partial y$ describes how *f* changes as *y* changes (holding *x* constant).
- If *f* is a function of three variables, *x*, *y*, and *z*, then the differential of *f* is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \,. \tag{2}$$

- We often write the partial derivatives with subscripts indicating which variables are held constant,

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z} dx + \left(\frac{\partial f}{\partial y}\right)_{x,z} dy + \left(\frac{\partial f}{\partial z}\right)_{x,y} dz \,,$$

  though it is not absolutely necessary to do so.
- That partial and full derivatives are different can be illustrated by dividing Eq. (1) by the differential of *x* to get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{3}$$

- From Eq. (3) we see that the full derivative and the partial derivative are equivalent only if *x* and *y* are independent, so that dy/dx is zero.
- ***WARNING!*** Partial derivatives are not like fractions. The numerators and denominators cannot be pulled apart or separated arbitrarily. Partial derivatives must be treated as a complete entity. So, you should **NEVER** pull them apart as shown below

$$\frac{\partial f}{\partial t} = axt^2 \quad \Rightarrow \quad \partial f = axt^2 \partial t \,.$$ ***NEVER DO THIS!***

With a full derivative this is permissible, because is it composed of the ratio of two differentials. But there is no such thing as a *partial differential*, ∂f .

THE CHAIN RULE

- If x and y are not independent, but depend on a third variable such as s [i.e., $x(s)$ and $y(s)$], then the chain rule is

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}. \tag{4}$$

- If x and y depend on multiple variables such as s and t [i.e., $x(s,t)$ and $y(s,t)$], then the chain rule is

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\end{aligned}. \tag{5}$$

THE PRODUCT RULE AND THE QUOTIENT RULE

- The product and quotient rules also apply to partial derivatives:
  - o The *product rule*

$$\frac{\partial}{\partial x}(uv) \equiv u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}. \tag{6}$$

  - o The *quotient rule*

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) \equiv \frac{1}{v^2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right). \tag{7}$$

PARTIAL DIFFERENTIATION IS COMMUTATIVE

- Another important property of partial derivatives is that it doesn't matter in which order you take them. In other words

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

- Multiple partial derivatives taken with respect to different variables are known as *mixed* partial derivative.

OTHER IMPORTANT IDENTITIES

- The reciprocals of partial derivatives are:

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial f}\right)_y} \quad ; \quad \left(\frac{\partial f}{\partial y}\right)_x = \frac{1}{\left(\frac{\partial y}{\partial f}\right)_x}$$

- If a function of two variables is constant, such as $f(x, y) = c$, then its differential is equal to zero,

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0 . \tag{8}$$

  - In this case, *x* and *y* must be dependent on each other, because in order for *f* to be a constant, as *x* change *y* must also change. For example, think of the function

$$f(x, y) = x^2 + y = c . \tag{9}$$

  - Eq. (8) can be rearranged to

$$\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dy} + \left(\frac{\partial f}{\partial y}\right)_x = 0 . \tag{10}$$

  The derivative dx/dy in Eq. (10) is actually a partial derivative with *f* held constant, so we can write

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_f + \left(\frac{\partial f}{\partial y}\right)_x = 0 ,$$

  which when rearranged leads to the identity

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial y}{\partial f}\right)_x \left(\frac{\partial x}{\partial y}\right)_f = -1 . \tag{11}$$

  - Eq. (11) is only true if the function *f* is constant, so that $df = 0$.

INTEGRATION OF PARTIAL DERIVATIVES

- Integration is the opposite or inverse operation of differentiation.

$$\int_a^b \frac{\partial f(s,t)}{\partial s} ds = f(b,t) - f(a,t)$$
$$\int_a^b \frac{\partial f(s,t)}{\partial t} dt = f(s,b) - f(s,a) \tag{12}$$

DIFFERENTIATING AN INTEGRAL

- If an integration with respect to one variable is then differentiated with respect to a separate variable, such as

$$\frac{\partial}{\partial t}\int_a^b f(s,t,u)ds$$

the result depends on whether or not the limits of integration, *a* and *b*, depend on *t*.

- In general, if both *a* and *b*, depend on *t*, the result is

$$\frac{\partial}{\partial t}\int_{a(t,u)}^{b(t,u)} f(s,t,u)ds = \int_{a(t,u)}^{b(t,u)} \frac{\partial f(s,t,u)}{\partial t} ds + f(b,t,u)\frac{\partial b}{\partial t} - f(a,t,u)\frac{\partial a}{\partial t}. \tag{13}$$

  - If *a* does not depend on *t* then the term in Eq. (13) that involves $\partial a/\partial t$ will disappear. Likewise, if *b* does not depend on *t*, then the term containing $\partial b/\partial t$ will be zero.