(((Existence Theorem)))

نبرهن وجود الحل ϕ بالنسبة للمعادلة التفاضلية

$$\left.\begin{array}{l} y' = f(t,y) \\ \phi(t_0) = y_0 \end{array}\right\} \quad \ldots (1)$$

Suppose f is continuous in region D that (t_0, y_0) point in D

observe that ivp (1) is equivalent to the problem of finding continuous function $y(t)$ satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds \quad \ldots (2)$$

EX: Determine the integral equation equivalent to the ivp. $y' = t^2 + y^4$, $y(0) = 1$

$$\int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} s^2 + y^4(s)\,ds$$

$$[y(s)]_{t_0}^{t} = \int_{t_0}^{t} [s^2 + y^4(s)]\,ds$$

$$y(t) = 1 + \int_{0}^{t} (s^2 + y^4(s))\,ds$$

Picard method:

هي طريقة تكرارية تحول المعادلة التفاضلية الى تكاملية وايجاد الحل بهذه الطريقة التكرارية

the in.p $y' = f(t,y)$
$y(t_0) = y_0$

the equivalent integral equation: $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$

we will solve the integral equation by successive approximation (Picard method) to $y(t)$.

the initial approximation $y(t_0) = y_0$ then define the sequence $y_1(t), y_2(t), \ldots, y_n(t)$

by $y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))\,ds$

$y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))\,ds$

$\vdots$

$y_n(t) = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,ds$

Such that $\lim_{n\to\infty} y_n(t) = y(t)$ كلما زادت التقريبات يقترب الحلول الى الحل المضبوط $y(t)$.

EX: Solve the equation; (Picard method)

$$y'(t) = 2t(1+y)$$

$$y(0) = 0$$

Solution: $y_1(t) = 0 + \int_0^t 2s(1+0)\,ds = \int_0^t 2s\,ds = t^2$

$y_2(t) = 0 + \int_0^t 2s(1+s^2)\,ds = s^2 + \frac{s^4}{2}\Big]_0^t = t^2 + \frac{t^4}{2}$

$y_3(t) = 0 + \int_0^t 2s(1+s^2 + \frac{s^4}{2})\,ds = s^2 + \frac{s^4}{2} + \frac{s^6}{6}\Big]_0^t = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$

$\vdots$

$y_n = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \cdots + \frac{t^{2n}}{n!}$

$\lim_{n\to\infty} y_n(t) = e^{t^2} - 1$

Ex: Solve the i.v.p. : $y' = ty+1$, $y(0)=0$ by the method of successive approximation

Solution:

$$y_1(t) = 0 + \int_0^t (s\,y(s)+1)\,ds$$

$$= \int_0^t s(0)+1\,ds = s\Big]_0^t = t$$

$$y_2(t) = \int_0^t (s^2+1)\,ds = \frac{s^3}{3}+s\Big]_0^t = \frac{t^3}{3}+t$$

$$y_3(t) = \int_0^t \left(s\left(\frac{s^3}{3}+s\right)+1\right)ds = \int_0^t \frac{s^4}{3}+s^2+1\,ds$$

$$= \frac{t^5}{3\cdot5}+\frac{t^3}{3}+t$$

$$\vdots$$

$$y_n(t) = \frac{t}{1}+\frac{t^3}{3}+\frac{t^5}{3\cdot5}+\cdots+\frac{t^{2n-1}}{3\cdot5\cdots(2n-1)}$$

H.W: Find Picard approximation of I.V.P

$y' = -ty+5$

$y(t) = 2$

Lipschitz condition:

Definition: A function f which satisfies an inequality of the form: $|f(t,y_2)-f(t,y_1)| \leq k\,|y_2-y_1|$ --- (3)
for all $(t,y_1),(t,y_2)$ in a region R:
$R=\{(t,y) \mid |t-t_0|<a,\ |y-y_0|<b\}$ is said to satisfy a Lipschitz condition in R.

<u>Ex</u>: If $f(t,y)=y^{1/3}$ in the rectangle $R=\{(t,y) \mid |t|\leq 1, |y|\leq 2\}$ then f does not satisfy a Lipschitz condition in R.

<u>solution</u>: let $(t,y_1),(t,0)$ two points for which eq.(3) fails to hold with any constant k

لكي اثبت: نختار زوج من النقاط بحيث لا تحقق شرط
لبشتز لذلك نقول انه المعادلة التفاضلية لا تحقق الشرط

$$\frac{f(t,y_1)-f(t,0)}{y_1-0} = \frac{y_1^{1/3}}{y_1} = y_1^{-2/3}$$

نختار $0 < y_1$ صغيره جداً
حيث $k = y_1^{-2/3}$ $\Leftarrow$ k كبيرة جداً $\Leftarrow$ لا يحقق الشرط

<u>ملاحظة</u>: لماذا ثابت لبشتز ؟؟
الجواب: وحدانية الحل
الوحدانية مهمة في المسائل التطبيقية لان وجود اكثر من حل واحد لا يفيد في الجوانب التطبيقية لهذا نفرض شرط على الدالة f يسمى بشرط لبشتز ويضمن لنا وحدانية الحل والدوال التي تحقق هذا الشرط تسمى دوال لبشتز.

Ex: $f(x,y) = x^2y^2 \quad \forall (x,y) \in D$

where $D = \{(x,y) : |x| \le 2, |y| \le 1\}$

Solution: $|f(x,y_1) - f(x,y_2)| = |x^2y_1^2 - x^2y_2^2| = x^2|y_1^2 - y_2^2|$

$\le x^2(|y_1| + |y_2|)(|y_1 - y_2|)$

$\le 4(1+1)(|y_1 - y_2|)$

$\le 8|y_1 - y_2|$ يحقق ثابت لبشتز

$k = 8$

Ex: $f(x,y) = \sin(xy) + e^{y^2}$

where $D = \{(x,y) : |x| \le 2, |y| \le 1\}$

prove that $f(x,y)$ satisfy Lipschitz condition

Solution:

المشتقة بالنسبة الى y

$f_y = x\cos(xy) + 2ye^{y^2}$

$|f(x,y_1) - f(x,y_2)| = |x\cos(xy_2) + 2y_2e^{y_2^2}| \cdot |y_1 - y_2|$

$\le [2(1) + 2(1)e^1]|y_1 - y_2|$

$\le 2(1+e)|y_1 - y_2|$

$\therefore L = 2(1+e)$ ثابت لبشتز

Lemma 1 : If ϕ is a solution of the i.v.p (1) on an interval I. Then ϕ satisfies (2) on I. Conversely if $y(t)$ is a solution of (2) on some interval J containing t_0 ; then $y(t)$ satisfies (1) on J and satisfy initial condition of (1)

proof: if ϕ solution of (1)

$$\phi'(t) = f(t, \phi(t))$$

integrating from t_0 to any t on I

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} f(s, \phi(s))ds$$

conversely :

If $y(t)$ is a continuous solution of (2)

differentiable (2)

$$y'(t) = f(t, y(t))$$

putting $t = t_0$ in (2) $\Rightarrow y(t_0) = y_0$

Now to establish the existence of solution, let us define the successive approximation in the general case by the equations

$$\left.\begin{aligned} \phi_0(t) &\equiv y_0 \\ \phi_{j+1}(t) &= y_0 + \int_{t_0}^{t} f(s, \phi_j(s))ds, \end{aligned}\right\} \quad j = 0, 1, 2, \dots \qquad (4)$$

Lemma 2: Define α to be the smaller of the positive number a and b/M. Then the successive approximations ϕ_j given by (4) are defined on the interval I given by $|t-t_0|<\alpha$ and on this interval

$$|\phi_j(t)-y_0| \le M|t-t_0| < b \qquad j=0,1,2,\dots \quad \dots(5)$$

Proof:

the proof is by induction.

$j=0$: $\phi_0(t)$ is defined on I and satisfies (5)

assume for $j=n\geq 1$ defined and satisfies (5)

$$\phi_0(t)=y_0 \qquad \dots(6)$$

$$\phi_{n+1}(t)=y_0+\int_{t_0}^{t} f(s,\phi_n(s))\,ds$$

to prove for $j=n+1$ that ϕ_{n+1} satisfies (5)

by (6): $|\phi_{n+1}(t)-y_0| = \left|\int_{t_0}^{t} f(s,\phi_n(s))\,ds\right| \le \left|\int_{t_0}^{t} |f(s,\phi_n(s))|\,ds\right|$

$\le M|t-t_0| < M\alpha \le b$

Since $|f(s,\phi_n(s))| \le M$ لأن f مقيدة

$\alpha = \min\{a, \frac{b}{M}\}$

Theorem: suppose f and $\frac{\partial f}{\partial y}$ are continuous and bounded on rectangle R and satisfy the bounds $|f(t,y)| \leq M$, $\left|\frac{\partial f}{\partial y}(t,y)\right| \leq K$. Then the successive approximations ϕ_j, given by (6), converge (uniformly) on interval $I: |t-t_0| < \alpha$, to a solution ϕ of the differential equation (1) and satisfies the initial condition $\phi(t_0) = y_0$.

Proof:

Define $r_j(t) = |\phi_{j+1}(t) - \phi_j(t)|$, $j = 0, 1, 2, \ldots$

$$r_j(t) = |\phi_{j+1}(t) - \phi_j(t)| = \left| \int_{t_0}^{t} [f(s,\phi_j(s)) - f(s,\phi_{j-1}(s))]\, ds \right|$$

$$\leq \int_{t_0}^{t} |f(s,\phi_j(s)) - f(s,\phi_{j-1}(s))|\, ds$$

by Lipschitz
$$\leq K \int_{t_0}^{t} |\phi_j(s) - \phi_{j-1}(s)|\, ds$$

$$= K \int_{t_0}^{t} r_{j-1}(s)\, ds, \quad j = 1, 2, \ldots \quad \text{---(7)}$$

the case $j=0$
$$r_0(t) = |\phi_1(t) - \phi_0(t)| = \left| \int_{t_0}^{t} f(s, \phi_0(s))\, ds \right|$$

$$\leq \int_{t_0}^{t} |f(s, \phi_0(s))|\, ds \leq M(t-t_0)$$

We will prove by induction that

$$r_j(t) \leq \frac{M K^{j} (t-t_0)^{j+1}}{(j+1)!} \quad \text{---(8)}$$

البرهان: الاستقراء الرياضي

عند $j=0$ نحصل r_0

نفرض $j=p-1$ صحيحة. نبرهن عند $p>1$

for $p>1$, by (7)

$$r_p(t) \le k\int_{t_0}^{t} r_{p-1}(s)\,ds \;\cancel{\le k\int r_{p-1}(s)\,ds}\; \le k\int_{t_0}^{t} \frac{Mk^{p-1}(s-t_0)^p}{p!}\,ds$$

$$= \frac{Mk^p(t-t_0)^{p+1}}{(p+1)!}, \quad t_0<t<t_0+\alpha$$

which is (8) for $j=p$

برهنا هذه الصيغة (8) من جهة اليمين
أي عند $t_0 < t < t_0+\alpha$
بنفس الطريقة نبرهنها عند $t_0-\alpha < t < t_0$
نحصل على ;

$\cancel{r_j(t) \le Mk^j}$

$$r_j(t) \le \frac{Mk^j|t-t_0|^{j+1}}{(j+1)!} = \frac{M[k|t-t_0|]^{j+1}}{k(j+1)!} < \frac{M(k\alpha)^{j+1}}{k(j+1)!}$$

$j=0,1,2,\dots$

$|t-t_0|<\alpha$

the series $\frac{M}{k}\sum_{j=0}^{\infty}\frac{(k\alpha)^{j+1}}{(j+1)!}$

converges to $\frac{M}{k}(e^{k\alpha}-1)$

by the comparison test the series $\sum_{j=0}^{\infty} r_j(t)$ converges on $|t-t_0|<\alpha$, this implies the absolute convergence on $|t-t_0|<\alpha$ of the series $\sum_{j=0}^{\infty}[\phi_{j+1}(t)-\phi_j(t)]$

since $\phi_j(t) = \phi_0(t) + \sum_{m=0}^{j-1}[\phi_{m+1}(t)-\phi_m(t)]$

then the sequence $\{\phi_j(t)\}$ convergence for all t in I

Now, we will show that this function $\phi(t)$ is continuous and satisfies the integral equation (2).

From the definition of $\phi(t)$

$$\phi(t) = \phi_0(t) + \sum_{n=0}^{\infty} (\phi_{n+1}(t) - \phi_n(t))$$

therefore, $\phi(t) - \phi_j(t) = \sum_{n=j}^{\infty} (\phi_{n+1}(t) - \phi_n(t))$

$$|\phi(t) - \phi_j(t)| \le \sum_{n=j}^{\infty} |\phi_{n+1}(t) - \phi_n(t)| \le \sum_{n=j}^{\infty} r_n(t)$$

$$\le \frac{M}{K} \sum_{n=j}^{\infty} \frac{(K\alpha)^{n+1}}{(n+1)!} \le \frac{M(K\alpha)^{j+1}}{K(j+1)!} \sum_{n=0}^{\infty} \frac{(K\alpha)^n}{n!}$$

$$= \frac{M}{K} \frac{(K\alpha)^{j+1}}{(j+1)!} e^{K\alpha}$$

let $\epsilon_j = \frac{(K\alpha)^{j+1}}{(j+1)!} \to 0$ as $j \to \infty$

To prove the continuity of $\phi(t)$ on I, let $\epsilon > 0$ be given.

we have: $\phi(t+h) - \phi(t) = \phi(t+h) - \phi_j(t+h) + \phi_j(t+h) - \phi_j(t)$
$+ \phi_j(t) - \phi(t)$

thus:

$$|\phi(t+h) - \phi(t)| \le \underbrace{|\phi(t+h) - \phi_j(t+h)|}_{\epsilon} + |\phi_j(t+h) - \phi_j(t)|$$
$$+ \underbrace{|\phi_j(t) - \phi(t)|}_{\epsilon} \le 2\epsilon_j + |\phi_j(t+h) - \phi_j(t)|$$

choosing j sufficiently large and $|h|$ sufficiently small and $\lim_{j\to\infty} \epsilon_j = 0$ and continuity of the $\phi_j(t)$

then $|\phi(t+h) - \phi(t)| < \epsilon$

to show that $\phi(t)$ satisfy the integral equation (2)

$$\left|\int_{t_0}^{t}[f(s,\phi(s))-f(s,\phi_j(s))]\,ds\right| \le k\left|\int_{t_0}^{t}|\phi(s)-\phi_j(s)|\,ds\right|$$

$$\le k\,\frac{M}{k}\,\underbrace{\frac{(k\alpha)^{j+1}}{(j+1)!}}_{\epsilon}\,e^{k\alpha}\,\underbrace{\left|\int_{t_0}^{t}ds\right|}_{|t-t_0|}$$

$$\le \epsilon_j\,\frac{M}{k}\cdot e^{k\alpha}\,|t-t_0|\cdot k$$

$$\le \epsilon_j\,\frac{M}{k}\,e^{k\alpha}\,\alpha k$$ this approaches zero as $j\to\infty$

for every t on I. this complete the proof