Lecture (8)

Convergence, Consistency and Stability

Preface: While numerically solving of a PDE we replace each derivative term by a Taylor series. Rearranging the terms gives us the equivalent difference formulation + the higher order terms called truncation error terms. The truncation error terms are to be neglected and we actually solve the equivalent difference equation instead of the PDE. The way we discretize the PDE (i.e. whether we approximate the derivatives using forward, backward or central differences etc.) gives rise to different schemes like FTCS (Forward Time Centered Space), Lax Wendroff etc. Three important characteristics a scheme must have are following:

CONVERGENCE: means that the solution to the finite difference approximation approaches the true solution of the PDE when the mesh is refined.

CONSISTENCY: A finite difference approximation is considered consistent if by reducing the mesh and time step size, the truncation error terms could be made to approach zero. In that case, the solution to the difference equation would approach the true solution to the PDE.

STABILITY: A finite difference approximation is stable if the errors (truncation, round-off, etc.) decay as the computation proceeds from one marching step to the next. Stability of a finite difference approximation is assessed using Von-Neumann stability analysis.

The important *Lax Equivalence Theorem* says that a finite difference approximation for a properly posed PDE satisfying consistency (meaning truncation error approaches zero when time step size and mesh size goes to zero) and stability (meaning error goes on diminishing from time step to time step) has necessary and sufficient conditions for convergence**.**

8.1 Criteria of Optimal Numerical Solution

The numerical solution can be very close to the exact solution of the partial differential equations if many criteria meet.

Firstly, the finite-difference analog in space and time must **converge** to its differential expression when the terms in the analog approach zero. For example, if $\Delta N/\Delta x$ is the finite difference analog of $\partial N/\partial x$, then the following convergence criterion:

$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \left\| \frac{\Delta N}{\Delta x} \right\| \tag{8.1}$$

must be available to say that the approximation is converged to its differential expression.

Secondly, the finite difference analog must be **consistent**.

- The finite-difference analog $\Delta N/\Delta x$ in equation (8.1) can be extracted from Taylor series expansion. In the expansion, the higher order terms are neglected to reduce the arithmetic burden of approximation. Difference between full Tayler series expansion and truncated approximation is the truncation error.
- The finite difference approximation for a derivative is *consistent* if the truncation error of the approximation approaches zero. We can say the consistence can be occurred when:

$$\lim_{\Delta x \to 0} \left\| TE\left(\frac{\Delta N}{\Delta x}\right) \right\| = 0 \qquad (8.2)$$

where TE is the truncation error of the approximation $\Delta N/\Delta x$.

Thirdly, if the finite difference approximation is consistent, then the rate that its truncation error approaches zero depends on the order of approximation. The *order of approximation* is the order of the term of the lowest rank in the neglected Tayler series expansion in the approximation.

The higher the order of approximation, the faster the truncation error approaches zero within a given spatial (or temporal) resolution.

- Thus with the same Δx, the high-order approximation is more accurate than the lower-order approximation.
- For the same truncation error, a low-order approximation requires smaller Δx than a high order approximation.
- As a result, the high order approximation with high Δx can has the same truncation error of the low order approximation with small Δx.

Fourth, while the analogs of the individual finite differences should converge towards exact differentials, the total numerical solution of PDE should converge to the exact solution when the spatial or temporal differences decrease towards zero. If we consider $N_{e,x,t}$ is exact solution, and $N_{f,x,t}$ is a finite difference solution of PDE, then the overall convergence occurs when:

$$\lim_{\Delta x, \Delta t \to 0} \left\| N_{e,x,t} - N_{f,x,t} \right\| = 0 \qquad (8.3)$$

If the finite difference solution is nonconvergent then it is not useful.

Fifth, for the numerical method to be successful, it must be stable. **Stability** occurs when the absolute value difference between numerical and exact solutions (error) does not grow with time. Accordingly,

$$\lim_{t\to\infty}\left\|N_{e,x,t} - N_{f,x,t}\right\| \le C \qquad (8.4)$$

where C is a constant.

- Stability often depends on the time-step size used. If the numerical solution is stable for any time step less than a certain value, then the solution is conditionally stable (as in the explicit method). If the solution is stable, regardless of the time step, then it is unconditionally stable (as in the implicit method). If the solution is unstable, regardless of the time step, then it is unconditionally unstable.
- The consistence and convergence of the individual analogues does not guarantee stability. On the other hand, stability is guaranteed if the scheme is generally convergent and its analogues for the finite differences are convergent and consistent.

8.2 Exercises

Q1. Discuss numerical convergency.

Q2. Discuss numerical stability.

Q3. Discuss numerical consistency.

Q4. Define order of approximation.

Q5. Define truncation error.

Q6. Discuss the relationship between the truncation error and order of appr-oximation.

Q7. Why do we truncate Taylor Expansion though we know that this will lessen the accuracy?

8.3 Homework: Summaries the section (8.1) in short and simplified manner.