Lecture (10)

Semi-Lagrangian Advection Scheme (Part 1)

10.1 The Semi-Lagrangian Scheme

The **Semi-Lagrangian scheme** (SLS) is a numerical method that is widely used in numerical weather prediction models for the integration of the equations governing atmospheric motion. A Lagrangian description of a system (such as the atmosphere) focuses on following individual air parcels along their trajectories as opposed to the Eulerian description, which considers the rate of change of system variables fixed at a particular point in space. A semi-Lagrangian scheme uses Eulerian framework but the discrete equations come from the Lagrangian perspective.

10.2 The Difference between the Eulerian and Lagrangian Flow

There are two ways to describe the fluid flow:

1. Eulerian Flow: *Stay at your place and watch the flow*!
   We refer to the rate of change of velocity with time at fixed point by the Eulerian derivative (or partial):
   $$\frac{\partial V}{\partial t} \qquad x, y, and\ z\ fixed$$
   Let us talk about the Eulerian (or local) change in details:
   Stand on a bridge, hang a thermometer inside the stream. The temperature you measure is at a fixed location. The change in temperature is local, and is given by the partial time derivative:
   $$\frac{\partial T}{\partial t} \qquad Change\ at\ a\ fixed\ location.$$
2. Lagrangian Flow: *Drift with the flow and see where you go*!
   We call the change in velocity with time following the flow by the Lagrangian derivative (or material or total):
   $$\frac{dV}{dt} \qquad parcel\ of\ fluid\ fixed$$

Now, get on a boat, and make the thermometer hanging in the river. The temperature you measure is at a point that moves with the stream. The change in temperature is given by the total time derivative:

$$\frac{dT}{dt} \qquad Change\ for\ a\ material\ parcel$$

Question: What is the relationship between $\frac{\partial V}{\partial t}$ and $\frac{dV}{dt}$?

Sol. Velocity is a function of space and time:

$$V = V(x(t), y(t), z(t), t)$$

The total change, following the stream, is given by the chain rule:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial V}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$= \frac{\partial V}{\partial t} + u \cdot \frac{\partial V}{\partial x} + v \cdot \frac{\partial V}{\partial y} + w \cdot \frac{\partial V}{\partial z}$$

$$= \frac{\partial V}{\partial t} + \vec{V} \cdot \vec{\nabla} V$$

This rule is true for all variables.

- It was found that the Eulerian leapfrog system is conditionally stable.
- Also, we know that stability condition is CFL criterion : $\frac{c\Delta t}{\Delta x} \leq 1$

- For high spatial resolution (small Δx) this severely limits the maximum time step Δt that is allowed.
- In NWP, time of the forecast is very important.
- In this lecture, we will study an alternative approach to time integration, which is unconditionally stable and free from restrictions of the CFL condition.

10.2 The Basic Idea of Semi-Lagrangian Scheme

(1) The semi-Lagrangian scheme for advection is based on the idea of approximating the Lagrangian time derivative.

(2) It is so formulated that the numerical domain of dependence always includes the physical domain of dependence. This necessary condition for stability is satisfied automatically by the scheme.

(3) In a fully Lagrangian scheme, the trajectories of actual physical parcels of fluid would be followed throughout the motion.

(4) The problem with this approach is that the distribution of representative parcels rapidly becomes highly non-uniform.

(5) In the semi-Lagrangian scheme the individual parcels are followed only for a single time-step. After each step, we revert to a uniform grid.

(6) The semi-Lagrangian algorithm has enabled us to integrate the primitive equations using a time step of 15 minutes. This can be compared to a typical time step of 2.5 minutes for conventional schemes.

(7) The consequential saving of computation time means that the operational numerical guidance is available to the forecasters much earlier than would otherwise be the case.

(8) Semi-Lagrangian advection schemes are now in widespread use in all the main Numerical Weather Prediction centers.

10.3 The Eulerian and the Lagrangian Approach

We consider the *linear advection equation* which describes the conservation of a quantity Y(x, t) following the motion of a fluid flow in one space dimension with constant advecting velocity c.

This may be written in either of two alternative forms:

$$\frac{\partial Y}{\partial t} + c\frac{\partial Y}{\partial x} = 0 \Leftarrow \ Eulerian\ Form$$

$$\frac{dY}{dt} = 0 \Leftarrow Lagrangian\ Form$$

The general solution is Y = Y (x − ct). (*Prove that*)

To develop numerical solution methods, we may start from either the Eulerian or the Lagrangian form of the equation. For the semi-Lagrangian scheme, we choose the latter.

Since the advection equation is linear, we can construct a general solution from Fourier components

$$Y = a\exp[ik(x - ct)]\,; \qquad k = 2\pi/L$$

This expression may be separated into the product of a function of space and a function of time:

$$Y = a \times \exp(-i\omega t) \times \exp(ikx)\,; \qquad \omega = kc$$

Therefore, in analyzing the properties of numerical schemes, we seek a solution of the form

$$Y_{p,q} = a \times \exp(-i\omega q\Delta t) \times \exp(ikp\Delta x) = aAq\ \exp(ikp\Delta x)$$

where $A = \exp(-i\omega\Delta t)$.

The character of the solution depends on the modulus of A:

If |A| < 1, the solution decays with time.

If |A| = 1, the solution is neutral with time.

If |A| > 1, the solution grows with time.

In the third case (growing solution), the scheme is unstable.