Real and Rational Numbers

1. Order Sets

(1.1) **Definition**: Let R be a relation on a set X, we say that

- R is a reflexive on X, if $xRx \ \forall x \in X$.
- R is a symmetric on X, if xRy, then yRx.
- R is a transitive on X, if xRy, and yRz, then xRz.
- R is an anti-symmetric on X, if xRy, and yRx, then $x = y$.

(1.2) **Note**: We say that R is a preorder relation on X, if R is a reflexive and transitive, and R is a partial order relation on X, if R is a reflexive, transitive and anti-symmetric.

(1.3) **Note**: Let $X \neq \emptyset$, then R is a partial order set, we say that (X, R) is a partially order set.

(1.4) **Example**: The order pairs $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathcal{R}, \leq)$, $(\mathbb{C}, \leq)$ where $(\leq = R)$ are a partially order sets.

(1.5) **Definition**: Let X be a partial order set, and $x, y \in X$ we say x, y are comparable, if $x \leq y$ or $y \leq x$.

(1.6) **Definition**: Let $A \subseteq X$, then A is a totally ordered or chain in X, if every two elements in X are comparable.

(1.7) **Example**: The order pairs $(\mathcal{R}, \leq)$, $(\mathbb{Q}, \leq)$are a totally ordered, but $(\mathbb{C}, \leq)$ does not.

(1.8) **Definition**: Let X be a partial order set and $a, b \in X$, we say that a is a first element or a smallest element in X, if $a \leq x \ \forall x \in X$. We say that b is a last element or greatest element in X, if $x \leq b \ \forall x \in X$.

(1.9) **Definition**: Let X be a partial order set, then X is called a well ordered, if every non empty subset of X contains a first element.

(1.10) **Definition**: Let X be a partial order set and $a, b \in X$, then a is called a minimal element in X, if $x \in X$, $x \leq a$, then $a = x$. We say that b is a maximal element in X, if $x \in X$, $b \leq x$, then $b = x$.

(1.11) **Examples**:

1. Let $A = \{-3, -2, -1, 0, 1, 4, 7\}$, then max $A = 7$ and min $A = -3$.
2. Min $\mathbb{N} = 1$ and max $\mathbb{N}$ does not exist.
3. Min $\mathbb{Z}$ and max $\mathbb{Z}$ does not exist.
4. Let $A = \{\frac{1}{n} : n \in \mathbb{Z}\}$, then max $A = 1$ and min A does not exist.
5. Let $A = \{-\frac{1}{n} : n \in \mathbb{Z}\}$, then min $A = -1$ and max A does not exist.
6. Let $A = \{\mp\frac{1}{n} : n \in \mathbb{Z}\}$, then max $A = 1$ and min $A = -1$.

(1.12) **Definition**: Let X be a partial order set and $A \subseteq X$, we say that $a \in X$ be lower bound of A, if $a \leq x \ \forall x \in A$. We say that a called a greatest lower bound of A, if its:

1. A lower bound of A;
2. $a' < a$ for all lower bound a' of A.

(1.13) **Note**: We denote of element which a greatest lower bound of A by inf A.

(1.14) **Definition**: Let X be a partial order set and $A \subseteq X$, we say that $b \in X$ be upper bound of A, if $x \leq b \ \forall x \in A$. We say that b called a smallest upper bound of A, if its:

3. An upper bound of A;
4. $b < b'$ for all upper bound b' of A.

(1.15) **Note**: We denote of element which a smallest upper bound of A by sup A.

(1.16) **Examples**:

1. Let $A = \{x \in \mathcal{R} : x \leq 2\}$, then sup $A = 2$ and inf A does not exist.
2. Let $A = \{x \in \mathcal{R} : -4 \leq x \leq 5\}$, then sup $A = 5$ and inf $A = -4$.

(1.17) **Definition**: Let X be a partial order set and $A \subseteq X$, we say that A is a bounded below, if there exist a lower bound and A is a bounded above, if there exists an upper bound. We say that A bounded, if A bounded from a lower and an upper.

(1.18) **Definition**: Let X be a partial order set. We say that X complete or complete ordered, if for all non empty subset and bounded from above A in X, then sup A exists.

2. **Real Numbers**

(2.1) **Axioms of Field**

1. **Axioms of abelian.**
   - $x + y = y + x \ \forall x, y \in \mathcal{R}$.
   - $x.y \quad = y.x \quad \forall x, y \in \mathcal{R}$.
2. **Axioms of associative.**
   - $x + (y + z) = (x + y) + z \ \forall x, y, z \in \mathcal{R}$.
   - $x.(y.z) \qquad = (x.y).z \qquad \forall x, y, z \in \mathcal{R}$.
3. **Axiom of distribution.**

$$x(y + z) = xy + xz \ \ \forall x, y, z \in \mathcal{R}$$

4. **Axioms of identity element.**
   - There is $0 \in \mathcal{R}$ such that $x + 0 = 0 + x = x$.
   - There is $1 \in \mathcal{R}$ such that $x.1 = 1.x = x$.

5. **Axioms of inverse element.**
   - For all $x \in \mathcal{R}$ there is $-x \in \mathcal{R}$ such that $x + (-x) = (-x) + x = 0$.
   - For all $x \in \mathcal{R}$, $x \neq 0$ there is $y \in \mathcal{R}$ such that $x.y = y.x = 1$.

(2.2) **Theorem:** Let $x, y, z \in \mathcal{R}$, then

1. $-(x - y) = y - x$.
2. $x - y = x + (-y)$.
3. $x + z = y + z$ iff $x = y$.
4. If $z \neq 0$, then $xz = yz$ iff $x = y$.
5. $xy = 0$ iff $x = 0$ or $y = 0$.
6. $(-x)y = x(-y) = -xy$.
7. $-(-x) = x$.
8. If $x \neq 0$, then $(-x)^{-1} = -x^{-1}$ and $(x^{-1})^{-1} = x$.

(2.3) **Axioms of order.**

There is a non-empty subset of $\mathcal{R}$ which denoted by $\mathcal{R}_+$ and its satisfy:

1. If $x, y \in \mathcal{R}_+$ then $x + y \in \mathcal{R}_+$ and $xy \in \mathcal{R}_+$.
2. If $x \in \mathcal{R}$ then one of following is true $-x \in \mathcal{R}_+$, $x = 0$, $x \in \mathcal{R}_+$.

(2.4) **Definition**:

1. If $x, y \in \mathcal{R}$ then $x < y$ if $y - x \in \mathcal{R}_+$.
2. $x \leq y$ means $x < y$ or $x = y$.
3. $x \leq y < z$ means $y < z$ and $x \leq y$.

(2.5) **Theorem:**

1. For all $x, y \in \mathcal{R}$ then either $x < y$ or $x > y$ or $x = y$.
2. If $x < y$ and $y < z$ then $x < z$.
3. $x + z < y + z$ iff $x < y$.
4. If $x < y$ and $z < w$ then $x + z < y + w$.
5. If $z > 0$ then $xz < yz$ iff $x < y$.
6. If $z < 0$ then $xz < yz$ iff $x > y$.
7. If $0 < x < y$ and $0 < z < w$ then $xz < yw$.

(2.6) **The Completeness Axiom.**

Let $\emptyset \neq A \subseteq \mathcal{R}$ then

1. If A is an upper bounded, then $\sup A$ exists.
2. If A is a lower bounded, then $\inf A$ exists.

(2.7) **Theorem:** Let $\emptyset \neq A \subseteq \mathcal{R}$ and $a, b \in \mathcal{R}$ then

1. $\operatorname{Inf} A = a$ iff
a. $a \leq x\ \forall x \in A$.
b. $\forall \varepsilon > 0\ \ \exists\, y \in A \ni \ \ y < a + \varepsilon$.
2. $\operatorname{Sup} A = b$ iff
a. $x \leq b\ \forall x \in A$.
b. $\forall \varepsilon > 0\ \ \exists\, y \in A \ni \ \ y > b - \varepsilon$.

Proof: Let $\inf A = a \Longrightarrow a$ is a lower bound of $A \Longrightarrow a \leq x\ \forall x \in A \Longrightarrow (a)$ satisfies.

Let $\varepsilon > 0 \Longrightarrow a + \varepsilon > a$, since a is greatest lower bound of $A \Longrightarrow a + \varepsilon$ not lower bound of $A \Longrightarrow \exists z \in A \ni z < a + \varepsilon \Longrightarrow (b)$ satisfies.

Now let (a), (b) are satisfy

(a) $\Longrightarrow a$ is a lower bound of A, let $c \in \mathcal{R} \ni a < c$. We must prove that c not lower of A. Put $\varepsilon = c - a \Longrightarrow \varepsilon > 0 \Longrightarrow \exists y \in A \ni y < a + \varepsilon \Longrightarrow y < a + (c - a) = c \Longrightarrow \inf A = a$

(2) Assume that $\sup A = b \Longrightarrow b$ an upper bound of $A \Longrightarrow x \leq b\ \forall x \in A \Longrightarrow (1)$

Now to prove (2) let $\varepsilon > 0 \Longrightarrow -\varepsilon < 0 \Longrightarrow b - \varepsilon < b$, since b is a smallest upper bound of $A \Longrightarrow b - \varepsilon$ does not upper bound of $A \Longrightarrow \exists z \in A \ni b - \varepsilon < y$.

(1) means b is an upper bound of A, let $d \in \mathcal{R} \ni d < b$. Put $\varepsilon = b - d \Longrightarrow \varepsilon > 0 \Longrightarrow \exists y \in A$ by (2) $\ni y > b - \varepsilon \Longrightarrow y > b - (b - d) = d \Longrightarrow \sup A = b$ ∎

(2.8) **Theorem:**(Archimedes property)

If $x, y \in \mathcal{R}$ and $x > 0$ then $\exists n \in \mathbb{Z}^+ \ni nx > y$.

Proof: Let $\exists a, b \in \mathcal{R} \ni a > 0$ and $na \leq b\ \forall\, n \in \mathbb{N}$. Put $A = \{na : n \in \mathbb{N}\}$, $1.a = a \in A \Longrightarrow \emptyset \neq A \subseteq \mathcal{R}$, $na \leq b\ \forall\, n \in \mathbb{N} \Longrightarrow b$ is an upper bound of $A \Longrightarrow A$ bounded from above. Since $\mathcal{R}$ satisfies the completeness $\Longrightarrow \exists y \in \mathcal{R} \ni y = \sup\ A$. $a > 0 \Longrightarrow -a < 0 \Longrightarrow y - a < y$ since y is a smallest upper bound of $A \Longrightarrow y - a$ does not upper bound of $A \Longrightarrow \exists m \in \mathbb{N} \ni y - a \leq ma \Longrightarrow y \leq ma + a \Longrightarrow y \leq (m+1)a$, since $m + 1 \in \mathbb{N} \Longrightarrow (m+1)a \in A \Longrightarrow y$ does not upper bound of $A \Longrightarrow$ contradiction ∎

(2.9) **Corollary**:

1. $\forall x \in \mathcal{R}_+ \exists\ n \in \mathbb{Z}_+ \ni \frac{1}{n} < x$.
2. $\forall x \in \mathcal{R} \exists\ n \in \mathbb{Z}_+\ \ni n > x$.
3. $\forall x \in \mathcal{R} \exists\ m, n \in \mathbb{Z}\ \ni m < x < n$.
4. $\forall x \in \mathcal{R} \exists$ a unique integer $n \in \mathbb{Z}\ \ni n \leq x < n + 1$.

Proof: (1) Put $b = 1$ and $a = \varepsilon \Longrightarrow \exists n \in \mathbb{Z}_+\ \ni na > b \Longrightarrow n\varepsilon > 1 \Longrightarrow \frac{1}{n} < \varepsilon$.

(2) Put $b = x$ and $a = 1 \Longrightarrow \exists n \in \mathbb{Z}_+\ \ni na > b \Longrightarrow n > x$.

(3) since $x \in \mathcal{R} \Longrightarrow$ by (2) $\exists n \in \mathbb{Z}_+\ \ni n > x$, now we must prove that $\exists m \in \mathbb{Z}_+\ \ni m < x$. Put $A = \{k \in\ \mathbb{Z}: k > -x\} \Longrightarrow A \subseteq \mathcal{R}$ and A is a lower bound (since $\mathcal{R}$ satisfies completeness) $\Longrightarrow \exists y \in \mathcal{R} \ni$ inf $A = y \Longrightarrow y > -x \Longrightarrow -y < x$, put $m = -y \Longrightarrow m < x$.

(4) Put $A = \{m \in\ \mathbb{Z}: m \leq x\} \Longrightarrow \emptyset \neq A \subseteq \mathcal{R}$ and A has an upper bound, (since $\mathcal{R}$ satisfies completeness) $\Longrightarrow \exists n \in \mathcal{R} \ni$ sup $A = n \Longrightarrow n \leq x$. To prove$x < n + 1$, suppose that $n + 1 \leq x \Longrightarrow n + 1 \in A$, but this is contradiction since, sup $A = n$ ■

3. **Field of Rational Numbers**

(3.1) **Theorem:** Every ordered field contains a field similar a field of rational number.

Proof: Let $(F, +, .)$ be ordered field. $n.1 = 1 + 1 + \cdots + 1$ (n-times), to prove if $n.1 = 0 \Longrightarrow n = 0$, let $k.1 = 0$ ($k \in \mathbb{Z}_+$), since $k.1 = 1 + 1 + \cdots + 1$(k-times)

$\Longrightarrow k > 1 \Longrightarrow k - 1 > 0 \Longrightarrow (k - 1).1 > 0 \Longrightarrow n.1 \in F\ \forall\, n \in \mathbb{Z}_+$ and $n.1 = 0$ iff $n = 0$, also $m.1 = n.1$ iff $m = n$. Since $(F, +, .)$ is a field $\Longrightarrow -(n.1) \in F \Longrightarrow -n.1 = (-1) + (-1) + \cdots + (-1)$ (n-times) $\Longrightarrow \mathbb{Z} \subset F$, since $(F, +, .)$ is a field $\Longrightarrow \forall n \in \mathbb{Z}, n \neq 0 \Longrightarrow \frac{1}{n} \in F \Longrightarrow \mathbb{Q} \subset F$. ■

(3.2) **Theorem:** the equation $x^2 = 2$ has no root in $\mathbb{Q}$.

Proof: Let $y \in \mathbb{Q} \ni y^2 = 2$, since $y \in \mathbb{Q} \Longrightarrow y = \frac{a}{b} \ni a, b \in \mathbb{Z}$, $b \neq 0$ and $g.c.d(a, b) = 1$. $y^2 = 2 \Longrightarrow \frac{a^2}{b^2} = 2 \Longrightarrow a^2 = 2b^2$ ... (1) $2b^2$is even number $\Longrightarrow a^2$ is even number $\Longrightarrow a$ is even number$\Longrightarrow a = 2c \Longrightarrow a^2 = 4c^2$, by (1) $\Longrightarrow 2b^2 = 4c^2 \Longrightarrow b^2$ is even number$\Longrightarrow b$ is even number $\Longrightarrow g.c.d(a, b) = 2$, but this is contradiction $\Longrightarrow y \notin \mathbb{Q}$ ■

(3.3) **Theorem:** the equation $x^2 = 2$ has an unique positive real root.

(3.4) **Corollary:** The field of rational numbers is a proper subset of a field of real numbers ($\mathbb{Q} \subset \mathcal{R}$).

Proof: Since $x^2 = 2$ has a root $\sqrt{2} \Rightarrow \sqrt{2} \in \mathcal{R}$, $x^2 = 2$ has no root in $\mathbb{Q} \Rightarrow \sqrt{2} \notin \mathbb{Q}$. ■

(3.5) **Theorem:** The field of rational numbers is an incomplete.

Proof: Let $A = \{x \in \mathbb{Q}: x^2 < 2\} \Rightarrow A \neq \emptyset$, let $y \in \mathbb{Q}$ with $\sup A = y \Rightarrow y^2 = 2$ or $y^2 < 2$ or $y^2 > 2$.

(1) $y^2 \neq 2$,

(2) If $y^2 < 2$, put $z = \frac{4+3y}{3+2y} \Rightarrow z \in \mathbb{Q}$, $z^2 - 2 = \left(\frac{4+3y}{3+2y}\right)^2 - 2 = \frac{y^2-2}{(3+2y)^2} < 0$, $(y^2 < 2) \Rightarrow z^2 < 2 \Rightarrow z \in A \Rightarrow z - y = \frac{4+3y}{3+2y} - y = \frac{2(2-y^2)}{3+2y} > 0 \Rightarrow z > y$, this is contradiction, since y is an upper bound of A.

(3) If $y^2 > 2 \Rightarrow z^2 > 2 \Rightarrow z$ is an upper bound of A, this is contradiction, since y is a smallest upper bound of A. ■

(3.6) **Theorem**(**Density of Rational Numbers**)

If $a, b \in \mathcal{R} \ni a < b \; \exists \, r \in \mathbb{Q} \ni a < r < b$.

Proof: (1) $b - a > 1$, put $A = \{n \in \mathbb{N}: n > a\}$, since $a \in \mathcal{R} \Rightarrow$ by Archimedes theorem $\Rightarrow \exists m \in \mathbb{N} \ni m > a \Rightarrow m \in A \Rightarrow A \neq \emptyset$. Since $\mathbb{N}$ is a well ordered and $\emptyset \neq A \subset \mathbb{N} \Rightarrow A$ contains a smallest number k, since $k \in A \Rightarrow k > a$, since k is a smallest number in $A \Rightarrow k - 1 \notin A \Rightarrow k - 1 \leq a \Rightarrow k \leq a + 1$, since $b - a > 1 \Rightarrow b > a + 1 \Rightarrow k < b \Rightarrow a < k < b \Rightarrow k \in \mathbb{Q}$.

(2) If $a < 0 < b \Rightarrow 0 \in \mathbb{Q}$.

(3) If $a < b < 0 \Rightarrow 0 < -b < -a$, by (1) $\exists \, r \in \mathbb{Q} \ni -b < r < -a \Rightarrow a < -r < b \Rightarrow -r \in \mathbb{Q}$. ■