3. Sequences

(3.1) **Definition**: Let X be a non-empty set. A function which its domain $\mathbb{N}$ and its codomain X is called a sequence in X, such that if $f: \mathbb{N} \longrightarrow X, \forall n \in \mathbb{N} \exists x_n \in X \ni f(n) = x_n$.

(3.2) **Example:** If $\{x_n\}$ be a sequence defined in $\mathcal{R} \ni x_n = (-1)^n \forall n \in \mathbb{N} \Longrightarrow \{x_n\} = \{(-1)^n\} = \{-1,1,-1,1,\dots\}$ a range is $\{x_n: n \in \mathbb{N}\} = \{-1,1\}$.

(3.3) **Definition**: Let $\{x_n\}, \{y_n\}$ be a sequences in X, we say that $\{y_n\}$ is a subsequence of $\{x_n\}$, if there is a function $\varphi: \mathbb{N} \longrightarrow \mathbb{N} \ni$

1. $x_n \circ \varphi = y_n$;
2. $\forall n \in \mathbb{N} \exists k \in \mathbb{N} \ni \varphi(m) \geq n \, \forall m \geq k$.

(3.4) **Example:** Let $x_n = \frac{1}{n}, \sigma_n = \frac{1}{2n-1}$, we note that $\{\sigma_n\}$ is a subsequence of $\{x_n\}$, since if we define $\psi: \mathbb{N} \longrightarrow \mathbb{N}$ by $\psi(n) = 2n - 1$, $\sigma_n = x_n \circ \psi = \frac{1}{2n-1}$ and then $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ be a subsequence of $\{\frac{1}{n}\}$.

(3.5) **Note:** If $\{y_n\}$ is a subsequence of $\{x_n\}$ and $\{y_n\}$ is a subsequence of $\{z_n\}$, then $\{z_n\}$ is a subsequence of $\{x_n\}$.

(3.6) **Definition**: If $\{x_n\}$ be a sequence in a partially ordered set X, we say that $\{x_n\}$ be an increasing, if $x_n \leq x_{n+1} \forall n$, and we say that $\{x_n\}$ be a decreasing, if $x_{n+1} \leq x_n \forall n$ and we say that $\{x_n\}$ be a monotone, if $\{x_n\}$ an increasing or a decreasing.

(3.7) **Note:**

- $x_n \uparrow = \{x_n\}$ be an increasing.
- $x_n \uparrow x = \{x_n\}$ be an increasing and $x_n = \sup x_n, n \in \mathbb{N}$.
- $x_n \downarrow = \{x_n\}$ be a decreasing.
- $x_n \downarrow x = \{x_n\}$ be a decreasing and $x_n = \inf x_n, n \in \mathbb{N}$.

(3.8) **Definition**: Let $\{x_n\}$ be a sequence in a partially ordered set X, we say that $\{x_n\}$ is a converges to $x \in X$, if there is $\{a_n\}, \{b_n\}$ in X, such that

1. $a_n \leq x_n \leq b_n \forall n$;
2. $a_n \uparrow x$ and $x_n \downarrow x_n$.

(3.9) **Note:** x is called a converge point and written $x_n \overset{0}{\rightarrow} x$.

(3.10) **Definition**: Let $\{x_n\}$ be a sequence in a partially ordered set X, we have

- Inferior limit = lim inf x_n, where lim inf $x_j, n \in \mathbb{N}, j \geq n$.
- Superior limit = lim sup x_n, where lim sup $x_j, n \in \mathbb{N}, j \geq n$.

(3.11) **Note:** If lim sup x_n = lim sup $x_n = x \Longrightarrow x_n \overset{0}{\rightarrow} x$.

Real Sequences

(3.12) **Note:** We say that $\{x_n\}$ be a real sequence if $X = \mathcal{R}$.

(3.13) **Definition**: The numerical sequence is a sequence which be subtract output of every term from direct previous term is equal to constant called progression basis and denoted by d.

(3.14) **Example:** The numerical sequence which its first term a and its basis d is

$\{a, a+d, a+2d, \dots, a+(n-1)d, \dots\}$. The general term of a numerical sequence $\{x_n\}$ is $x_n = a + (n-1)d$ where a represents a first term and d represents a basis with the partial summation

$$S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} (a + (k-1)d) = \frac{n}{2}(2a + (n-1)).$$

(3.15) **Definition**: Geometry progression is a sequence which output of division of every term on direct previous term is equal to a constant called progression basis and denoted by r.

(3.16) **Example:** Geometric progression which its first term a and its basis r is

$\{a, ar, ar^2, \dots, ar^n, \dots\}$. The general term is $x_n = ar^{n-1}$ where a represents a first term and r represents a basis with the partial summation

$$S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}, r \neq 1.$$

If $r = 1 \Longrightarrow S_n = a + a + \cdots + a = na$.

If $|r| < 1 \Longrightarrow \sum_{k=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

(3.17) **Definition**: Arithmetic geometric progression is $\{a, (a+d)r, (a+2d)r^2, \dots, (a+(n-1)d)r^{n-1}, \dots\}$. The general term is $x_n = (a+(n-1)d)r^{n-1}$ and the partial summation is

$S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n}(a+(k-1)d)r^{k-1} = \frac{a(1-r^n)}{1-r} + \frac{rd(1-nr^{n-1})+(n-1)r^n}{(1-r)^2}, r \neq 1.$

If $|r| < 1 \Rightarrow \sum_{k=1}^{\infty}(a+(n-1)d)r^{n-1} = \frac{a}{1-r} + \frac{rd}{(1-r)^2}$.

(3.18) **Definition**: Let $\{x_n\}$ be a real sequence, we say that $\{x_n\}$

1. Convergent, if $\exists\, r \in \mathcal{R} \ni \forall \varepsilon > 0\, \exists k \in \mathbb{Z}^+ \ni |x_n - x| < \varepsilon\ \forall n > k,$ we say that a point x is a limit point of $\{x_n\}$ and its written by $\lim_{n\to\infty} x_n$ or $x_n \to x$ where $n \to \infty$, therefore $x_n \to x$ iff $|x_n - x| \to 0$.
2. Divergent, if $\{x_n\}$ does not convergent.
3. Cauchy sequence, if $\forall \varepsilon > 0 \exists\ k \in \mathbb{Z}^+ \ni |x_n - x_m| < \varepsilon\ \forall n, m > k$ and then $\{x_n\}$ is a Cauchy sequence iff $|x_n - x_m| \to 0$ where $n, m \to \infty$.

(3.19) **Examples:**

1. Show that $\{x_n\} \to x$.

Solution: since $\forall \varepsilon > 0 \Rightarrow |x_n - x| = 0 < \varepsilon$.

2. Show that $\{\frac{1}{n}\} \to 0$.

Solution: since $\forall \varepsilon > 0$ (by Archimedes property), $\exists k \in \mathbb{Z}^+ \ni \frac{1}{k} < \varepsilon \Rightarrow \forall n > k \Rightarrow \frac{1}{n} < \frac{1}{k} \Rightarrow \frac{1}{n} < \varepsilon$, so $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon\ \forall n > k$.

3. Show that $\{n\}$ be a divergent.

Solution: since if we assume that $\{n\}$ be a convergent $\Rightarrow \exists x \in \mathcal{R} \ni\ x_n \to x$ and then $\forall \varepsilon > 0 \Rightarrow (x-\varepsilon, x+\varepsilon)$ contains of terms $\{n\}$, since $x + \varepsilon \in \mathcal{R} \Rightarrow$ (by Archimedes property) $\Rightarrow \exists k \in \mathbb{Z}^+ \ni x + \varepsilon < k$, since $x + \varepsilon < k < k+1 < \cdots \Rightarrow k, k+1, k+2\ \notin (x-\varepsilon, x+\varepsilon)$, this means $(x-\varepsilon, x+\varepsilon)$ does not contain on terms of $\{n\}$, but this is contradiction.

4. Show that $\{x_n\}$ such that $x_n = \begin{cases} n, & n \leq 10^6 \\ 1, & n > 10^6 \end{cases}$ converges to one .

Solution: since $\forall \varepsilon > 0$, take $k > 10^6\ \forall n > k \Rightarrow x_n = 1$ and then $|x_n - 1| = 0 < \varepsilon \Rightarrow x_n \to 1$.

5. Show that $\{(-1)^n\}$ be a divergent.

Solution: since if we suppose that $\{(-1)^n\}$ be a convergent $\Longrightarrow \exists x \in \mathcal{R} \ni x_n = (-1)^n \rightarrow x$, let $\varepsilon > 0 \; \exists k \in \mathbb{Z}^+ \ni |x_n - x| < \varepsilon \; \forall n > k \Longrightarrow |(-1)^n - x| < \varepsilon \; \forall n > k \Longrightarrow x - \varepsilon < (-1)^n < x + \varepsilon \; \forall n > k \Longrightarrow (-1)^n \in (x - \varepsilon, x + \varepsilon) \forall n > k$.

Let $x = 1$, take $\varepsilon = \frac{1}{4} \Longrightarrow \frac{1}{4}\varepsilon > 0 \Longrightarrow (-1)^n \in (1 - \frac{1}{4}\varepsilon, 1 + \frac{1}{4}\varepsilon) \forall n$ is an even, $(-1)^n \notin (1 - \frac{1}{4}\varepsilon, 1 + \frac{1}{4}\varepsilon) \forall n$ is an odd, this means that $(1 - \frac{1}{4}\varepsilon, 1 + \frac{1}{4}\varepsilon)$ does not contain all terms of $\{(-1)^n\}$ and then $\{(-1)^n\}$ does not converge to 1.

By same way we prove that $\{(-1)^n\}$ does not converge to -1.

Now, let $x \neq 1, x \neq -1$, let $a_1 = |1 - x|, a_2 = |-1 - x|$, take $\varepsilon \ni \varepsilon < a_1, \varepsilon < a_2$, we deduce that $(x - \varepsilon, x + \varepsilon)$ does not contain on any term of$\{(-1)^n\} \Longrightarrow (-1)^n$ does not converge to x.

(3.20) **Theorem:**

1. If a real sequence is a convergent, then a converge point is a unique.
2. Every convergent sequence be Cauchy sequence.

Proof: (1) Let $x_n \rightarrow x$, $x_n \rightarrow y \ni x \neq y$ and let $|x - y| = \varepsilon \Longrightarrow \varepsilon > 0$, since $x_n \rightarrow x \Longrightarrow \exists k_1 \in \mathbb{Z}^+ \ni |x_n - x| < \frac{\varepsilon}{2} \forall n > k_1$, $x_n \rightarrow y \Longrightarrow \exists k_2 \in \mathbb{Z}^+ \ni |x_n - y| < \frac{\varepsilon}{2} \forall n > k_2$ put $k =$max $\{k_1, k_2\} \Longrightarrow |x_n - x| < \frac{\varepsilon}{2}, |x_n - y| < \frac{\varepsilon}{2} \; \forall n > k \Longrightarrow \varepsilon = |x - y| = |(x_n - x) + (x_n - y| \leq |x_n - x| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, but this is a contradiction $\Longrightarrow x = y$.

(2) let $\{x_n\}$ be a convergent sequence $\Longrightarrow \exists x \in X \ni x_n \rightarrow x$, let $\varepsilon > 0$, since $x_n \rightarrow x \Longrightarrow \exists k \in \mathbb{Z}^+ \ni |x_n - x| < \frac{\varepsilon}{2} \forall n > k$, if $n, m \geq k \Longrightarrow |x_n - x_m| = |(x_n - x) + (x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and then $\{x_n\}$ be Cauchy sequence.

(3.21) **Definition**: If $\{x_n\}$ be a real sequence, we say that $\{x_n\}$ is

1. Bounded above, if $\exists M_1 \in \mathcal{R} \ni x_n \leq M_1 \forall n$;
2. Bounded below, if $\exists M_2 \in \mathcal{R} \ni M_2 \leq x_n \forall n$;
3. Bounded, if $\exists M \in \mathcal{R} \ni |x_n| \leq M \; \forall n$.

(3.22) **Examples**:

1. $\{\frac{1}{n}\}$ is a bounded, since $\left|\frac{1}{n}\right| < 2\ \forall n.$
2. $\{\frac{n}{n+1}\}$ is a bounded, since $\left|\frac{n}{n+1}\right| < 1\ \forall n.$
3. $\{(-1)^n\}$ is a bounded, since $|(-1)^n| \leq 1\ \forall n.$
4. $\{n\}$ does not bounded, since if we suppose that $\{n\}$ is a bounded $\Longrightarrow \exists M \in \mathcal{R}^+ \ni |n| \leq M\ \ \forall n$, but this is a contradiction (Archimedes property) since $n \geq M, n \in \mathbb{Z}^+$.
5. $\{3^n\}$ does not bounded.

(3.23) **Theorem**: Every Cauchy sequence be a bounded, and then every convergent sequences be a bounded.

Proof: Let $\{x_n\}$ be Cauchy sequence, we must prove that $\{x_n\}$ is a bounded. Let $\varepsilon = 1$, since $\{x_n\}$ is a Cauchy sequence $\Longrightarrow \exists k \in \mathbb{Z}^+ \ni |x_n - x_m| < 1 \forall n, m > k$, let $m = k + 1 \Longrightarrow |x_n - x_m| < 1 \forall n > k$, since $|x_n| - |x_{k+1}| \leq |x_n - x_{k+1}| \Longrightarrow |x_n| - |x_{k+1}| < 1\ \ \forall n > k \Longrightarrow |x_n| < 1 + |x_{k+1}|\ \forall n > k$, put $M = \max\{|x_1|, |x_2|, \ldots, |x_k|, |x_{k+1}| + 1\}$, and then $\{x_n\}$ is a bounded.

(3.24) **Note**: If a real sequence is a bounded, then its not a necessary be a convergent, for example $\{(-1)^n\}$ is a bounded, but does not convergent.

(3.25) **Definition**: Let $\{x_n\}$ be a real sequence. We said that $\{x_n\}$

1. Non-decreasing, if $x_n \leq x_{n+1}\ \forall n.$
2. Increasing, if $x_n < x_{n+1}\ \forall n.$
3. Non-increasing, if $x_{n+1} \leq x_n\ \forall n.$
4. Decreasing, if $x_{n+1} < x_n\ \forall n.$

(3.26) **Note**: We said that $\{x_n\}$ is a monotonic, if its be satisfy any one of above.

(3.26) **Examples**:

1. $\{\frac{1}{\sqrt{2}}\}$ is a decreasing $\Longrightarrow$ a monotonic.
2. $\{\frac{n}{n+1}\}$ is an increasing $\Longrightarrow$ a monotonic.
3. $\{(-1)^n\}$ does not a monotonic.

(3.27) **Theorem**:

1. Every bounded real sequence and monotonic be a convergent.
2. Every bounded real sequence contains on a convergent partial sequence.

(3.28) **Theorem**: (Some special sequences)

1. If $p > 0 \Longrightarrow x_n = \frac{1}{n^p} \to 0$.
2. If $p > 0 \Longrightarrow x_n = n^p \to 1$.
3. $x_n = \sqrt[n]{n} \to 1$.
4. If $|a| < 1 \Longrightarrow x_n = a^n \to 0$.

Proof: (1) let $\varepsilon > 0$, take $k > (\frac{1}{\varepsilon})^{1/p}$ $\forall n > k \Longrightarrow n > (\frac{1}{\varepsilon})^{1/p} \Longrightarrow \frac{1}{n^p} < \varepsilon \Longrightarrow \left|\frac{1}{n^p} - 0\right| < \varepsilon$.

(2) a. if $p > 1 \Longrightarrow \sqrt[n]{p} > 1$, put $y_n = \sqrt[n]{p} - 1 \Longrightarrow y_n > 0 \Longrightarrow \sqrt[n]{p} = 1 + y_n \Longrightarrow (1 + y_n)^n = 1 + ny_n + \frac{n(n-1)}{2}{y_n}^2 + \cdots + {y_n}^n$, since $y_n > 0\ \forall n \Longrightarrow p \geq 1 + ny_n \Longrightarrow \frac{p-1}{n} \geq y_n \Longrightarrow 0 < y_n < \frac{p-1}{n} \Longrightarrow y_n \to 0 \Longrightarrow x_n \to 0$.

b. if $p = 1 \Longrightarrow \sqrt[n]{p} = 1\ \forall n \Longrightarrow \sqrt[n]{p} \to 1$.

c. if $0 < p < 1 \Longrightarrow \frac{1}{p} > 1$, put $\lambda = \frac{1}{p} \Longrightarrow \sqrt[n]{p} = \frac{1}{\sqrt[n]{\lambda}}$ and $\sqrt[n]{\lambda} \to 1$, since $\lambda > 0 \Longrightarrow \sqrt[n]{p} \to 1$.

(3) let $y_n = \sqrt[n]{n} - 1$, since $\sqrt[n]{n} > 1 \forall n \Longrightarrow y_n > 0\ \forall n \Longrightarrow \sqrt[n]{n} = 1 + y_n \Longrightarrow n = (1 + y_n)^n = 1 + ny_n + \frac{n(n-1)}{2}{y_n}^2 + \cdots + {y_n}^n \Longrightarrow n > \frac{n(n-1)}{2}{y_n}^2 \Longrightarrow {y_n}^2 < \frac{2}{n-1} \Longrightarrow |y_n| < \sqrt{\frac{2}{n-1}} \Longrightarrow y_n < \sqrt{\frac{2}{n-1}}\ \forall n \geq 2 \Longrightarrow y_n \to 0 \Longrightarrow x_n \to 0$.

(4) $\forall \varepsilon > 0\ \exists k \in \mathbb{Z}^+ \ni |x_n| < \varepsilon\ \forall n > k$.

a. if $a = 0, k = 0$.

b. if $a \neq 0 \Longrightarrow \frac{1}{|a|}$ exists, put $b = |a| - 1 \Longrightarrow \frac{1}{|a|} = 1 + b$, since $|a| < 1 \Longrightarrow \frac{1}{|a|} > 1 \Longrightarrow b > 0$, $|a^n| = |a|^n = \frac{1}{(1+b)^n}$, $(1 + b)^n = 1 + nb + \cdots + b^n > nb \Longrightarrow \frac{1}{(1+b)^n} < \frac{1}{nb} \Longrightarrow |a^n| < \frac{1}{nb}\ \forall n$, put $k > \frac{1}{\varepsilon b} \forall n > k \Longrightarrow n > \frac{1}{\varepsilon b} \Longrightarrow \frac{1}{nb} < \varepsilon \Longrightarrow |a^n| < \varepsilon \Longrightarrow x_n = a^n \to 0$.

(3.29) **Theorem**: Let $\{x_n\}, \{y_n\}$ be a real sequences such that $x_n \to x$ and $y_n \to y$, then

1. $x_n + y_n \to x + y$.
2. $\lambda x_n \to \lambda x \ \forall \lambda \in \mathcal{R}$.
3. $\lambda + x_n \to \lambda + x \ \forall \lambda \in \mathcal{R}$.
4. $x_n y_n \to xy$.
5. $\frac{x_n}{y_n} \to \frac{x}{y}$ where $y_n \neq 0 \ \forall n$.
6. $\frac{1}{y_n} \to \frac{1}{y}$ where $y_n \neq 0 \ \forall n$.
7. $|x_n| \to |x|$.
8. $|x_n - y_n| \to |x - y|$.
9. If $x_n \leq y_n \Longrightarrow x \leq y \ \forall n$.

Proof: (1) let $\varepsilon > 0$, since $x_n \to x \Longrightarrow \exists k_1 \in \mathbb{Z}^+ \ni |x_n - x| < \frac{\varepsilon}{2} \forall n > k_1$, since $y_n \to y \Longrightarrow \exists k_2 \in \mathbb{Z}^+ \ni |y_n - y| < \frac{\varepsilon}{2} \forall n > k_2$, put $k =$ max $\{k_1, k_2\} \Longrightarrow |x_n - x| < \frac{\varepsilon}{2}, |y_n - y| < \frac{\varepsilon}{2} \forall n > k$, $|(x_n - x) + (x_n - y)| \leq |x_n - x| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $x_n + y_n \to x + y$.

(3.30) **Theorem**:

1. For all real number, there is Cauchy sequence of rational numbers converge of them.
2. For all real number, there is Cauchy sequence of irrational numbers converge of them.
3. There is Cauchy sequence of rational numbers does not converge to any rational number.

Proof: (1) let $r \in \mathcal{R}$, since $r - \frac{1}{n} < r < r + \frac{1}{n} \forall n \in \mathbb{Z}^+ \Longrightarrow$ (by density of rational numbers) $\Longrightarrow \forall n \in \mathbb{Z}^+ \exists r_n \in \mathbb{Q} \ni |r_n - r| < \frac{1}{n} \ \forall n \in \mathbb{Z}^+ \Longrightarrow r - \frac{1}{n} < r_n < r + \frac{1}{n} \forall n \in \mathbb{Z}^+$, now, we must prove that $r_n \to r$, let $\varepsilon > 0 \Longrightarrow$ (by Archimedes property) $\Longrightarrow \exists k \in \mathbb{Z}^+ \ni \frac{1}{k} < \varepsilon \ \forall n > k \Longrightarrow \frac{1}{n} < \frac{1}{k} \Longrightarrow |r_n - r| < \frac{1}{n} < \frac{1}{k} < \varepsilon \Longrightarrow r_n \to r$.

(3.31) **Definition**: We said that a space X is a complete, if every Cauchy sequence in X be a convergent in X.

(3.32) **Note**: $\mathbb{Q}$ is an incomplete, while $\mathcal{R}$ is a complete.