5. A Converging Test

(5.1) **Theorem**: If $\sum_{n=1}^{\infty} a_n$ be a convergent, then $a_n \to 0$.

Proof: let $\varepsilon > 0, S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$, since $\sum_{n=1}^{\infty} a_n$ is a convergent $\Longrightarrow \{S_n\}$ is a convergent and then $\{S_n\}$ is Cauchy sequence $\Longrightarrow \exists k \in \mathbb{Z}^+ \ni |S_n - S_m| < \varepsilon \forall\, n, m > k$, if $n > k \Longrightarrow S_n = a_1 + a_2 + \cdots + a_n$, $S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$, $S_{n+1} - S_n = a_{n+1}$, $|a_{n+1} - 0| = |a_{n+1}| = |S_{n+1} - S_n| < \varepsilon \Longrightarrow a_n \to 0$.

(5.2) **Note**: If $a_n \to 0 \Longrightarrow \sum_{n=1}^{\infty} a_n$ not necessary be a convergent, for example $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent, while $a_n = \frac{1}{n} \to 0$.

(5.3) **Corollary:**

1. If $\lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} a_n$ is a divergent.
2. If $\sum_{n=1}^{\infty} a_n$ is a convergent and $a_n \neq 0\ \forall n \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{a_n}$ is a divergent.

(5.4) **Example**: Show that $\sum_{n=1}^{\infty} \frac{n^2+2n+3}{3n^2-1}$ be a divergent.

$a_n = \frac{n^2+2n+3}{3n^2-1} \Longrightarrow a_n \to \frac{1}{3} \neq 0 \Longrightarrow \sum_{n=1}^{\infty} \frac{n^2+2n+3}{3n^2-1}$ is a divergent.

(5.5) **Definition**: We say that $\sum_{n=1}^{\infty} b_n$ be a dominate on $\sum_{n=1}^{\infty} a_n$, if $|a_n| \leq b_n \forall n$.

(5.6) **Theorem:** Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be a non-negative of terms and $\sum_{n=1}^{\infty} b_n$ is a dominate on $\sum_{n=1}^{\infty} a_n \ni a_n \leq b_n \forall n$ or

1. If $\sum_{n=1}^{\infty} b_n$ is a convergent $\Longrightarrow \sum_{n=1}^{\infty} a_n$ is a convergent.
2. If $\sum_{n=1}^{\infty} a_n$ is a divergent $\Longrightarrow \sum_{n=1}^{\infty} b_n$ is a divergent.

Proof; (1) let $S_n = \sum_{k=1}^{n} a_k, T_n = \sum_{k=1}^{n} b_k$, since $\sum_{n=1}^{\infty} b_n$ is a convergent $\Longrightarrow \{T_n\}$ is a convergent and $\{T_n\}$ is a bounded $\Longrightarrow \exists M > 0 \ni |T_n| \leq M\ \forall n$, since $0 \leq a_n \leq b_n \forall n \Longrightarrow 0 \leq \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k\ \forall n \Longrightarrow 0 \leq S_n \leq T_n \Longrightarrow 0 \leq S_n \leq M \Longrightarrow \{S_n\}$ is a bounded, since $\{S_n\}$ does not increasing and then $\{S_n\}$ is a convergent $\Longrightarrow \sum_{n=1}^{\infty} a_n$ is a convergent.

(2) let $\sum_{n=1}^{\infty} b_n$ is a convergent $\Longrightarrow \sum_{n=1}^{\infty} a_n$ is a convergent, but this is a contradiction $\Longrightarrow \sum_{n=1}^{\infty} b_n$ is a divergent.

(5.7) **Example:** Discuss a convergent of 1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ 2. $\sum_{n=1}^{\infty} \frac{1}{n+2}$.

(1) since $n^2 + 2n + 1 > n^2 + 2n \Rightarrow (n+1)^2 > n(n+1) \Rightarrow \frac{1}{(n+1)^2} < \frac{1}{n(n+1)} \forall n$, since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is a convergent.

(2) since $n + 2 \leq 3n \Rightarrow \frac{1}{n+2} \geq \frac{1}{3n} \forall n$, since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+2}$ is a divergent.

(5.8) **Example**: Let $a_n > 0\ \forall n$, if $\sum_{n=1}^{\infty} a_n$ is a convergent, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is a convergent.

$a_n > 0 \Rightarrow 1 + a_n > 1 \Rightarrow \frac{1}{1+a_n} < 1 \Rightarrow \frac{a_n}{1+a_n} < a_n$, since $\sum_{n=1}^{\infty} a_n$ is a convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is a convergent.

(5.9) **Definition**: Let $p \in \mathcal{R}$, we said $\sum_{n=1}^{\infty} \frac{1}{n^p}$ a p −series.

(5.10) **Theorem:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a convergent, if $p > 1$ and it's a divergent, if $p \leq 1$.

Proof; (1) if $p = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent.

(2) if $p < 1 \Rightarrow \frac{1}{n} \leq \frac{1}{n^p}$, since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergent$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergent where $p < 1$.

(3) if $p > 1$, we compare with geometry series such $a = 1, r = \frac{2}{2^p}$, since $p > 0 \Rightarrow r > 1 \Rightarrow \sum_{n=1}^{\infty} (\frac{2}{2^p})^{n-1}$ is a convergent, since $\frac{1}{n^p} < (\frac{2}{2^p})^{n-1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ is a convergent.

(5.11) **Theorem:** Let $a_n \geq 0, b_n \geq 0\ \forall n$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be both a convergent or a divergent.

(5.12) **Corollary:** Let $a_n \geq 0, b_n \geq 0\ \forall n$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is a convergent, then $\sum_{n=1}^{\infty} a_n$ is a convergent.

Ratio Test

(5.13) **Theorem:** Let $a_n \geq 0$ and $\{\frac{a_{n+1}}{a_n}\}$ converges to r (i.e. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$), then $\sum_{n=1}^{\infty} a_n$ be

1. A convergent, if $r < 1$.
2. A divergent, if $r > 1$.
3. Where $r = 1$, then the test will be failed.

Proof: (1) If $r < 1 \Longrightarrow \exists r \in \mathcal{R} \ni r < \lambda < 1$ (by density of rational number) $\Longrightarrow \lambda - r > 0$, since $\frac{a_{n+1}}{a_n} \to r \Longrightarrow \exists k \in \mathbb{Z}^+ \ni \left|\frac{a_{n+1}}{a_n} - r\right| < \lambda - r\ \forall n > k \Longrightarrow -(\lambda - r) < \frac{a_{n+1}}{a_n} - r < \lambda - r\ \forall n > k \Longrightarrow 2r - \lambda < \frac{a_{n+1}}{a_n} < \lambda \forall n > k$, let $n > k, \frac{a_{n+1}}{a_{k+1}} = \frac{a_{k+2}}{a_{k+1}} \cdot \frac{a_{k+3}}{a_{k+2}} \cdot \frac{a_{k+4}}{a_{k+3}} \ldots \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}} \cdot \frac{a_{n+1}}{a_n} < \lambda . \lambda \ldots \lambda = \lambda^{n-k} \Longrightarrow \frac{a_{n+1}}{a_{k+1}} < \lambda^{n-k} = \frac{\lambda^n}{\lambda^k} \Longrightarrow a_{n+1} < \frac{a_{k+1}}{\lambda^k}\lambda^n$, since $\lambda < 1 \Longrightarrow \sum_{n=1}^{\infty} \frac{a_{k+1}}{\lambda^k}\lambda^n$ is a convergent $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ is a convergent. (by ratio test).

(5.14) **Theorem:** Let $a_n \geq 0\ \forall n$, if $\exists k \in \mathbb{Z}^+, b < 1 \ni \frac{a_{n+1}}{a_n} < b\ \forall n > k \Longrightarrow \sum_{n=1}^{\infty} a_n$ is a convergent.

(5.15) **Example:** Discuss a convergent of (1) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (2) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ (3) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2^n}$.

(1) $a_n = \frac{2^n}{n!}$ and $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = 2\ \frac{1}{n+1} \Longrightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \Longrightarrow \sum_{n=1}^{\infty} \frac{2^n}{n!} \text{ is a convergent.}$$

(2) $a_n = \frac{2^n}{n^2}$ and $a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = 2\ (\frac{n}{n+1})^2 \Longrightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 \Longrightarrow \sum_{n=1}^{\infty} \frac{2^n}{n^2} \text{ is a divergent.}$$

(3) $a_n = \frac{\sqrt{n}}{2^n}$ and $a_{n+1} = \frac{\sqrt{n+1}}{2^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\sqrt{n+1}}{2^{n+1}}}{\frac{\sqrt{n}}{2^n}} = \frac{1}{2}\sqrt{\frac{n+1}{n}} \Longrightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} \Longrightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2^n} \text{ is a convergent.}$$

The Root Test

(5.16) **Theorem:** Let $a_n \geq 0\ \forall n$.

1. If $\exists k \in \mathbb{Z}^+, b < 1 \ni \sqrt[n]{a_n} < b\ \forall n > k \Rightarrow \sum_{n=1}^{\infty} a_n$ is a convergent.
2. If $\sqrt[n]{a_n} \geq 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is a divergent.

(1)Let $\exists k \in \mathbb{Z}^+, b < 1 \ni \sqrt[n]{a_n} < b\ \forall n > k \Rightarrow a_n < b^n \forall n > k,$ since $b < 1 \Rightarrow \sum_{n=1}^{\infty} b^n$ is a convergent$\Rightarrow \sum_{n=1}^{\infty} a_n$ is a convergent (by root test).

(5.17) **Example:** Discuss a convergent of (1) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ (2) $\sum_{n=1}^{\infty} \frac{4^n}{n^3}$

(1) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n} = 2\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ and $a_n = \frac{2^n}{n^n} \Rightarrow \sqrt[n]{a_n} = \frac{2}{n} \Rightarrow \sqrt[n]{a_n} \leq \frac{2}{3} \forall n > 2 \Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n^n}$ is a convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ is a convergent.

Integral Test

(5.18) **Theorem:** Let f be a continuous function which positive, decreasing and defined $\forall x \geq 1$, let $a_n = f(n) \forall n \in \mathbb{Z}^+ \Rightarrow \sum_{n=1}^{\infty} a_n$ is a convergent iff $\int_1^{\infty} f(x)dx$ is a convergent.

(5.19) **Example:** Does $\sum_{n=1}^{\infty} ne^{-n}$ convergent?

Solution: $a_n = ne^{-n} \forall n \geq 1 \Rightarrow f(x) = xe^{-x} \forall x \geq 1 \Rightarrow f$ is a continuous and positive. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0\ \forall x \geq 1 \Rightarrow f$ is a decreasing and $f(x) > 0 \forall x \geq 1 \Rightarrow f$ is a positive.

$\int_1^t f(x)dx = \int_1^t xe^{-x}dx = -e^{-x}(x+1)|_1^t = -\frac{t+1}{e^t} + \frac{2}{e}$.

$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t xe^{-x}dx = -0 + \frac{2}{e} = \frac{2}{e} \Rightarrow \sum_{n=1}^{\infty} ne^{-n}$ is a convergent.

Alternations Series

(5.20) **Definition**: We said that $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ is an alternation, if $a_n > 0\ \forall n$.

(5.21) **Theorem:** If $0 < a_{n+1} < a_n\ \forall n \in \mathbb{Z}^+,\ \lim_{n\to\infty} a_n \Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1} a_n$ and $\sum_{n=1}^{\infty}(-1)^n a_n$ is a convergent.

(5.22) **Definition**: We said that $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent, if $\sum_{n=1}^{\infty}|a_n|$ is a convergent.

(5.23) **Definition**: We said that $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent, if $\sum_{n=1}^{\infty} a_n$ is a convergent but $\sum_{n=1}^{\infty}|a_n|$ is a divergent.

(5.24) **Examples:**

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^{n-1}}$ is an absolutely convergent, since $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ is a convergent.
2. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}$ is a convergent, but $\sum_{n=1}^{\infty}\frac{1}{n}=\sum_{n=1}^{\infty}\left|(-1)^{n}\frac{1}{n}\right|$ is a divergent $\Longrightarrow$ $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}$ is a conditionally convergent.