## 6. Metric Spaces

(6.1) **Definition**: If $X$ be a non-empty set. We said that $d: X \times X \to \mathcal{R}$ is a metric function on $X$, if

1. $d(x, y) \geq 0 \;\; \forall x, y \in X$.
2. $d(x, y) = 0 \;\; \Leftrightarrow x = y$.
3. $d(x, y) = d(y, x) \;\; \forall x, y \in X$.
4. $d(x, y) \leq d(x, z) + d(z, y) \;\; \forall x, y, z \in X$.

(6.2) **Note**: $(X, d)$ is called Metric Space.

(6.3) **Example**: Let $d_u: \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is a function defined by $d_u(x, y) = |x - y| \;\; \forall x, y \in \mathcal{R}$, then $d_u$ is a metric function, and $(\mathcal{R}, d_u)$ is an usual metric space.

**Solution:** (1) let $x, y \in \mathcal{R} \Longrightarrow x - y \in \mathcal{R} \Longrightarrow |x - y| \geq 0 \Longrightarrow d_u(x, y) \geq 0$.

(2) $d_u(x, y) = 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

(3) let $x, y \in \mathcal{R}, d_u(x, y) = |x - y| = |y - x| = d_u(y, x)$.

(4) let $x, y, z \in \mathcal{R}, \;\; x - y = (x - z) + (z - y)$

$|x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y|$

$d_u(x, y) \leq d_u(x, z) + d_u(z, y) \Longrightarrow d_u$ is a metric function on $\mathcal{R}$.

(6.4) **Example**: Let $d: \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is a function defined by $d(x, y) = |x - y| + 1 \;\; \forall x, y \in \mathcal{R}$, does $d$ a metric function on $\mathcal{R}$?

**Solution:** let $x, y \in \mathcal{R} \ni x = y \Longrightarrow d(x, y) = 1$ this means the second axiom does not satisfy $\Longrightarrow d$ does not metric.

(6.5) **Example**: Let $X$ be a non-empty set and $d: X \times X \to \mathcal{R}$ is a function defined by $d(x, y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$ for all $x, y \in X$, then $d$ is a metric function and $(X, d)$ is called discrete metric space.

**Solution:**

(1) since $d(x, y) = 0$ or $d(x, y) = 1 \;\; \forall x, y \in X \Longrightarrow d(x, y) \geq 0 \;\; \forall x, y \in X$.

(2) let $x, y \in X$, if $x = y \Longrightarrow d(x, y) = 0$ (by the definition of a function $d$), if $d(x, y) = 0 \Longrightarrow x = y$, since if $x \neq y \Longrightarrow d(x, y) = 1$, but this is a contradiction $\Longrightarrow$ $d(x, y) = 0 \Longleftrightarrow x = y$.

(3) let $x, y \in X$, $d(x, y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases} = \begin{cases} 0, y = x \\ 1, y \neq x \end{cases} = d(y, x)$.

(4) let $x, y, z \in X$

a. if $x = y \Longrightarrow d(x, y) = 0$, since $d(x, z) \geq 0, d(z, y) \geq 0 \Longrightarrow d(x, z) + d(z, y) \geq 0 \Longrightarrow d(x, y) \leq d(x, z) + d(z, y)$.

b. if $x \neq y \Longrightarrow d(x, y) = 1$, so $z \neq x$ or $z \neq y$, let $z \neq x \Longrightarrow d(x, z) = 1$ since $d(z, y) \geq 0 \Longrightarrow d(x, z) + d(z, y) \geq 1 \Longrightarrow d(x, y) \leq d(x, z) + d(z, y) \Longrightarrow d$ is a metric function on $X$.

(6.6) **Example**: Let $X$ be a non-empty set and $d: X \times X \to \mathcal{R}$ is a function defined by $d(x, y) = 0$ for all $x, y \in X$, then $d$ is a metric function and $(X, d)$ is called indiscrete metric space.

(6.7)**Theorem:** Let $(X, d)$ be metric space, then

1. $|d(x, z) - d(z, y)| \leq d(x, y)\ \forall x, y, z \in X$.
2. $|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w) \forall x, y, z \in X$.

**Proof:**

(1) $d(x, z) \leq d(x, y) + d(y, z) = d(x, y) + d(z, y)$

$d(x, z) - d(z, y) \leq d(x, y)$ ... (1) Also

$d(z, y) \leq d(z, x) + d(x, y) = d(x, z) + d(x, y)$

$d(z, y) - d(x, z) \leq d(x, y)$

$-d(x, z) - d(z, y) \leq d(x, y)$

$d(x, z) - d(z, y) \geq -d(x, y)$ ... (2) $\Longrightarrow$ from (1), (2) $\Longrightarrow$

$-d(x, y) \leq d(x, z) - d(z, y) \leq d(x, y) \Longrightarrow |d(x, z) - d(z, y)| \leq d(x, y)$.

(6.8)**Theorem:** Let $X$ be a non-empty set, then a function $d: X \times X \to \mathcal{R}$ be a metric function iff

1. $d(x, y) = 0$ iff $x = y$.

2. $d(x,y) \leq d(x,z) + d(y,z)\ \forall x,y,z \in X$.

**Proof:** $\Longrightarrow$) suppose that $d$ is a metric function $\Longrightarrow (1),(2)$ are satisfy (from a definition).

$\Longleftarrow$) if $(1),(2)$ are satisfy,

(1)Let $x,y \in X$ from (2), we get

$d(x,x) \leq d(x,y) + d(x,y) = 2d(x,y)$, but $d(x,y) = 0$ from $(1) \Longrightarrow d(x,y) \geq 0$.

(2) same the condition (1).

(3) Let $x,y \in X$ from (2), we get

$d(y,x) \leq d(y,y) + d(x,y) = 0 + d(x,y) = d(x,y)$

$d(x,y) \leq d(x,x) + d(y,x) = 0 + d(y,x) = d(y,x) \Longrightarrow$

$d(x,y) \leq d(y,x), d(y,x) = d(x,y) \Longrightarrow d(x,y) = d(y,x)$.

(4) Let $x,y,z \in X$

$d(x,y) \leq d(x,y) + d(y,z) = d(x,z) + d(z,y) \Longrightarrow d$ is a metric function on $X$.

(6.9)**Definition:** Let $X$ be metric space and $\emptyset \neq A \subset X$. The diagonal of $A$ denoted by $\delta(A)$ and defined by $\delta(A) = \sup\{d(x,y): x,y \in A\}$, if $A = \emptyset$ or $A$ contains on only one element, then $\delta(A) = 0$. The distance of point $p$ from $A$ denoted by $d(p,A)$ and defined by $d(p,A) = \inf\{d(p,x): x \in A\}$.

(6.10)**Note:** Its clear, if $p \in A$, then $d(p,A) = 0$, and if $A = \emptyset$, then $d(p,\emptyset) = \infty$.

(6.11)**Definition:** Let $\emptyset \neq B \subset X$. The distance between $A,B$ is denoted by $d(A,B)$ and defined by $d(A,B) = \inf\{d(x,y): x \in A, y \in B\}$. Its clear that, if $A = \emptyset$, then $d(\emptyset,B) = \infty$.

(6.12)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space and $A = [1,2), B = (2,4]$, we note that $\delta(A) = 1, \delta(B) = 2,\ d\left(\frac{5}{4},A\right) = 0, d\left(\frac{3}{2},B\right) = \frac{1}{2}, d\left(\frac{9}{4},B\right) = 0, d(5,B) = 1, d(A,B) = 0$.

(6.13)**Theorem:** Let $(X,d)$ be metric space and $\emptyset \neq A \subset X$, then

1. $\delta \geq 0, d(p,A) \geq 0\ \forall p \in X, d(A,B) \geq 0$.

2. If $A$ is a finite, then $\delta(A) = \infty$.
3. If $A \cap B \neq \emptyset$, then $d(A, B) = 0$.
4. $|d(p, A) - d(q, A)| \leq d(p, q)\ \forall p, q \in X$.
5. If $A \subset B$, then $d(p, A) \geq d(p, B)\ \forall p \in X$.

**Proof:**

(1) and (2) from the definition.

(3) since $A \cap B \neq \emptyset \Longrightarrow \exists x_0 \in A \cap B \Longrightarrow x_0 \in A$ and $x_0 \in B$

$d(A, B) = \inf\{d(x, y): x \in A, y \in B\} \leq d(x, y)\ \forall x \in A,\ \forall y \in B$

$\Longrightarrow d(A, B) \leq d(x_0, x_0)$, but $d(A, B) \geq 0 \Longrightarrow d(A, B) = 0$

(6.14)**Notes:**

1. If $d(p, A) = 0$, then not necessary that $p \in A$.
2. If $d(A, B) = 0$, then not necessary that $A \cap B \neq \emptyset$.