8. Metric Topologies

(8.1) **Definition**: Let (X,d) be a metric space, $x_0 \in X$ and let $r \in \mathcal{R}^+$. The set $\{x \in X: d(x,x_0) < r\}$ is called an open ball in X, such x_0 is a center of a ball and r is radius of a ball and denoted by $B_r(x_0) = \{x \in X: d(x,x_0) < r\}$.

(8.2) **Definition**: A closed ball with center x_0 and radius r is denoted by $\overline{B_r}(x_0) = \{x \in X: d(x,x_0) \leq r\}$.

(8.3) **Example**: In usual metric space, we have

1. Every open ball contains an open interval.
2. Every closed ball contains a closed interval.

Solution: (1) $d(x,y) = |x-y| \ \forall x,y \in \mathcal{R}$.

Let $x_0 \in \mathcal{R},\ r > 0$

$B_r(x_0) = \{x \in X: d(x,x_0) < r\} = \{x \in X: |x-x_0| < r\}$

$= \{x \in X: -r < x - x_0 < r\} = \{x \in X: x_0 - r < x < x_0 + r\} = (x_0 - r, x_0 + r)$.

(8.4) **Example**: Let $X = [0,1]$ and a function $d: X \times X \to \mathcal{R}$ defined by $d(x,y) = |x-y| \ \forall x,y \in X$. Discuss $B_1\left(\frac{1}{2}\right)$ and $B_{\frac{1}{4}}(0)$.

Solution: $B_1\left(\frac{1}{2}\right) = \left\{x \in X: d\left(x,\frac{1}{2}\right) < 1\right\} = \left\{x \in X: \left|x - \frac{1}{2}\right| < 1\right\}$

$= \left\{x \in X: \frac{-1}{2} < x < \frac{3}{2}\right\} = \{x \in X: 0 \leq x \leq 1\} = X$.

(8.5) **Example**: Discuss an open balls with the center $(0,0)$ and radius 1 for following metric functions:

1. $d_1(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} \ \ \forall x = (x_1,x_2), y = (y_1,y_2) \in \mathcal{R}^2$.
2. $d_2(x,y) = |x_1-y_1| + |x_2-y_2| \ \ \forall x = (x_1,x_2), y = (y_1,y_2) \in \mathcal{R}^2$.
3. $d_3(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\} \ \ \forall x = (x_1,x_2), y = (y_1,y_2) \in \mathcal{R}^2$.

Solution: (1) $r = 1,\ \ (x_0,y_0) = (0,0)$

$B_r(x_0) = \{x \in X: d(x,x_0) < r\} = \{(x_1,x_2) \in \mathcal{R}^2: {x_1}^2 + {x_2}^2 < 1\}$.

(8.6) **Example**: Let (X,d) be discrete metric space and let $x_0 \in X, r \in \mathcal{R}^+$, then

1. If $r > 1$, then $B_r(x_0) = X$.

2. If $r \leq 1$, then $B_r(x_0) = \{x_0\}$.

Solution: (1) Let $x \in X$, since $d(x, x_0) = \begin{cases} 0, x = x_0 \\ 1, \ x \neq x_0 \end{cases}$

$\Rightarrow \ d(x, x_0) < r \Rightarrow x \in B_r(x_0) \Rightarrow X \subseteq B_r(x_0)$, but $B_r(x_0) \subseteq X \Rightarrow B_r(x_0) = X$.

(8.7) **Definition**: Let (X, d) be a metric space and $A \subseteq X$. We said that A is an open set in X, if $\forall x \in X \ \exists r > 0 \ni B_r(x) \subset A$.

(8.8) **Definition**: We say that A is a closed set in X, if A^c is an open set in X.

(8.9)**Theorem**: In any metric space, we have

1. Every open ball is an open set.
2. Every closed ball is a closed set.

Proof: (1) Let (X, d) a metric space and $x_0 \in X, r > 0$.

We must prove that $B_r(x_0)$ is an open set.

Let $x \in B_r(x_0) \Rightarrow d(x, x_0) < r \Rightarrow r - d(x, x_0) > 0$

Put $r - d(x, x_0) = r_1 \Rightarrow r_1 > 0$, we must prove $B_{r_1}(x_0) \subseteq B_r(x_0)$.

Let $y \in B_{r_1}(x_0) \Rightarrow d(y, x_0) < r_1 \Rightarrow d(y, x_0) < r - d(y, x_0) < r$

$\Rightarrow d(y, x) + d(y, x_0) < r$

Since $d(y, x_0) \leq \ d(y, x) + d(x, x_0) \Rightarrow d(y, x_0) < r \Rightarrow y \in B_r(x_0)$

$\Rightarrow B_r(x_0)$ is an open set.

(8.10)**Corollary:** In usual metric space $(\mathcal{R}, d_u)$, we have

1. Every an open interval is an open set.
2. Every a closed interval is a closed set.

(8.11)**Theorem**: Let (X, d) is a metric space and $A \subseteq X$, then A is an open iff A equals to union of an open balls.

Proof: If $A = \emptyset$, the proof will end.

If $A \neq \emptyset$, let A is an open set in X.

$\Longrightarrow \forall x \in A\ \exists r_x > 0 \ni B_{r_x}(x) \subseteq A \Longrightarrow A \subseteq \bigcup_{x\in A} B_{r_x}(x) \subset A$

$\Longrightarrow A = \bigcup_{x\in A} B_{r_x}(x) \Longrightarrow A$ equals to union of an open balls.

$\Longleftarrow$) let A equals to union of an open balls.

Since each open ball is an open set $\Longrightarrow A$ equals to union of an open set.

$\Longrightarrow A$ is an open set.

(8.12) **Example**: Prove that, every subset of discrete metric space is an open and closed.

Solution: Let (X, d) is discrete metric space and $A \subseteq X$. If $A = \emptyset$, the proof will end.

If $A \neq \emptyset$, let $x \in A$, take $r = \frac{1}{2}$.

$$B_r(x) = \left\{y \in X: d(y,x) < \frac{1}{2}\right\} = \{y \in X: d(y,x) = 0\} = \{y \in X: y = x\} = \{x\} \subset A$$

$\Longrightarrow A$ is an open set.

Let $B \subseteq X \Longrightarrow B^c \subseteq X \Longrightarrow B^c$ is an open set in $X \Longrightarrow B$ is a closed set.

(8.13)**Theorem**: Let (X, d) be a metric space.

1. Each of $\emptyset, X$ be an open sets in X.
2. If $A_1, A_2, \dots, A_n$ be an open sets in X, then $\bigcap_{i=1}^{n} A_i$ be an open set in X.
3. If $A_\lambda \forall \lambda \in \Lambda$ is an open set in X, then $\bigcup_{\lambda\in\Lambda} A_\lambda$ be an open set in X.

Proof: (1) suppose that $\emptyset$ be a non- open set

$\Longrightarrow \exists x \in \emptyset \ni B_r(x) \subseteq \emptyset\ \forall r > 0$, this is impossible, since $\emptyset$ does not contain on element $\Longrightarrow \emptyset$ is an open set.

Since $B_r(x) \subseteq X\ \forall x \in X, r > 0 \Longrightarrow X$ be an open set.

(8.14) **Example**: Let $(\mathcal{R}, d_u)$ be usual metric space, and let $A_n = \left(\frac{-1}{n}, \frac{1}{n}\right) \forall n \in \mathbb{Z}^+$, we note that A_n be an open set $\forall n \in \mathbb{Z}^+$ and $\bigcap_{i=1}^{\infty} A_n = \{0\}$ be a non- open set.

(8.15)**Theorem**: Let (X, d) be a metric space.

1. Each of $\emptyset, X$ be a closed sets in X.
2. If $A_1, A_2, \dots, A_n$ be a closed sets in X, then $\bigcup_{i=1}^{n} A_i$ be a closed set in X.
3. If $A_\lambda \forall\, \lambda \in \Lambda$ is a closed set in X, then $\bigcap_{\lambda \in \Lambda} A_\lambda$ be a closed set in X.

Proof: (1) since $\emptyset^c = X$ and X is an open set in $X \Longrightarrow \emptyset^c$ is an open set in X

$\Longrightarrow \emptyset$ is a closed set in X

Since $X^c = \emptyset$ and $\emptyset$ is an open set in $X \Longrightarrow X^c$ is an open set in X

$\Longrightarrow X$ is a closed set in X.

(8.16) **Example**: Let $(\mathcal{R}, d_u)$ be usual metric space, and let $A_n = \left[\frac{1}{n}, 1\right] \forall n \in \mathbb{Z}^+$, we note that A_n be a closed set $\forall n \in \mathbb{Z}^+$ and $\bigcup_{i=1}^{\infty} A_n = (0,1]$ be a non- closed set.

(8.17) **Notes**:

1. The point $x_0 \in A' \Longleftrightarrow \forall$ open ball with center x_0 contains on infinite number of points in A.
2. $\bar{A} = \{x \in X : d(x, A) = 0\}$.

(8.18)**Theorem**: In any metric space (X, d) be every single set is a closed, $\Longrightarrow$ every finite set be a closed.