15. Continuity and Compactness

(15.1)**Theorem**: Let $(X, d_1), (Y, d_2)$ are metric spaces and $f: X \to Y$ is a continuous function. If X is a compact set , then $f(X)$ is a compact set in Y.

Proof: let $F = \{G_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $f(X)$ in Y.

$$\Rightarrow f(X) \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda, G_\lambda \in \tau_\lambda \forall \lambda \in \Lambda$$

$$X \subseteq f^{-1}(f(X)) \subseteq f^{-1}(\bigcup_{\lambda \in \Lambda} G_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda)$$

Since $\bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda) \subseteq X \Rightarrow X = \bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda)$

Since f is a continuous $\Rightarrow f^{-1}(G_\lambda)$ is an open set in $X \ \forall \lambda \in \Lambda$

$\{f^{-1}(G_\lambda)\}$ is an open cover of X

Since X is a compact space $\Rightarrow \exists\ \lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda \ni X = \bigcup_{i=1}^{n} f^{-1}(G_\lambda)$

$$\Rightarrow X = f^{-1}(\bigcup_{i=1}^{n} G_\lambda) \Rightarrow f(X) = f(f^{-1}(\bigcup_{i=1}^{n} G_\lambda)) \subseteq \bigcup_{i=1}^{n} G_\lambda$$

$\Rightarrow f(X)$ is a compact set in Y.

(15.2)**Corollary**: Let $(X, d_1), (Y, d_2)$ are metric spaces and $f: X \to Y$ is a continuous function. If A is a compact set in X , then $f(A)$ is a compact set in Y.

(15.3)**Example:** Let $(\mathcal{R}, d_u)$ is usual metric space and $f: \mathcal{R} \to \mathcal{R}$ is defined as $f(x) = 2\ \forall x \in \mathcal{R}$.

We note that f is a continuous, since f is a constant and $A = \{1,2,3\}$ is a compact in $\mathcal{R}$, since A is a finite, but $f^{-1}(A) = \mathcal{R}$ does not compact.

(15.4)**Theorem**: Let $(X, d_1), (Y, d_2)$ are metric spaces $\ni X \cong Y$, then X is a compact space $\Leftrightarrow Y$ is a compact space.

Proof: since $X \cong Y \Rightarrow \exists\ f: X \to Y$.

Let X is a compact space, since f is a continuous $\Rightarrow f(X)$ is a compact in Y.

Since f is a bijective $\Rightarrow f(X) = Y \Rightarrow Y$ is a compact space.

Now, let Y is a compact space

Since $f^{-1}: Y \to X$ is a continuous $\Longrightarrow f^{-1}(Y) = X$ is a compact.

(15.5)**Theorem**: Let (X, d) is a compact space and $f: X \to \mathcal{R}$ is a continuous function, then

1. f is a bounded.
2. If $\alpha = \inf \{f(x): x \in X\}, \beta = \sup \{f(x): x \in X\}$, then $\exists\, a, b \in X \ni f(a) = \alpha, f(b) = \beta$.

Proof: (1) since X is a compact space and f is a continuous $\Longrightarrow f(X)$ is a compact set in $\mathcal{R}$.

Since every compact set in $\mathcal{R}$ is a closed and bounded $\Longrightarrow f(X)$ is a bounded.

(2) since $f(X)$ is a bounded $\Longrightarrow \exists \alpha, \beta \in \mathcal{R}$ and since $f(X)$ is a closed $\Longrightarrow \alpha, \beta \in f(X)$

Put $a \in f^{-1}(\{\alpha\}), b \in f^{-1}(\{\beta\}) \Longrightarrow f(a) = \alpha, f(b) = \beta$.

(15.6)**Theorem**: Let $(X, d_1), (Y, d_2)$ are a metric spaces and $f: X \to Y$ is a continuous function. If X is a compact space, then f is an uniform continuous.

(15.7)**Corollary**: Let $(\mathcal{R}, d_u)$ is usual metric space. If $f: [a, b] \to \mathcal{R}$ is a continuous function, then f is an uniform continuous.