$x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$

$x_i \geq 0 \quad (i = 1, 2, \ldots, 7)$

Solution to Linear Programing (LP)

Before going through the solution of (L.P) we need to know some definitions and concepts.

Solution; A set of real values of $X = (x_1, x_2, \ldots, x_n)$ which satisfies the constraint $AX (\leq = \geq) b$.

Feasible Solution:- A set of real values of X = (x1, x2, ..., xn) which Satisfies the constraints $AX(\leq = \geq)$ b and satisfies the non-negativity restriction $X \geq 0$.

Feasible Region: The collection of all the feasible solution is called as the feasible region.

When a LP is solved, one of the following cases will occur:

Case 1 The LP has a unique solution when has only one optimal solution

Case 2 The LP has more than one (actually an infinite number of) optimal solutions. This is the case of **alternative (Multiple) optimal solutions.**

Case 3 The LP is **infeasible** (it has no feasible solution). This means that the feasible region contains no points.

Case 4 The LP is unbounded. This means (in a max problem) that there are points in the feasible region with arbitrarily large z-values (objective function value).

There are many methods to solve Linear programing problem. Such as :

GRAPHICAL METHOD

If the objective function Z is a function of two variables, then the problem can be solved by graphical method.

A procedure of solving LPP by graphical method is as follows:

i. Formulation of the problem into LPP model (i.e) objective function and the constraints are written down.
ii. Consider each inequality constraints as equation.

iii. Plot each equation on the graph as each equation will geometrically represent a straight line.

iv. Shade the feasible region and identify the feasible solutions. Every point on the line will satisfy the equation of line. If the inequality constraints corresponding to that line is $\leq$ then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraints with $\geq$ sign the region above the line in the first quadrant is shaded. The point lying in common region will satisfy all the constraints simultaneously. Thus, the common region obtained is called **feasible region**. This region is the region of **feasible solution**. The corner points of this region are identified.

v. Finding the optimal solutions. The values of Z at various corners points of the region of feasible solution are calculated. The optimum (maximum or minimum) Z among these values is noted. Corresponding solution is the optimal solution.

Example: Unique Solution

Solve the following LPP graphically

Max $Z = 4x + 5y$

Subject to

$x + y \leq 20$

$3x + 4y \leq 72$

$x, y \geq 0$

SOLUTION

First, we need to draw the straight lines of constrains. This means finding x- intercept and y- intercept.

For the constrain $x + y \leq 20$

x- intercept (set $y = 0$) is

$x + 0 = 20$, then $x = 20$, the point (20, 0) y- intercept (set x = 0) is

0 + y = 20 , then y = 20 , the point is (0, 20)

For the constrain $3x + 4y \leq 72$

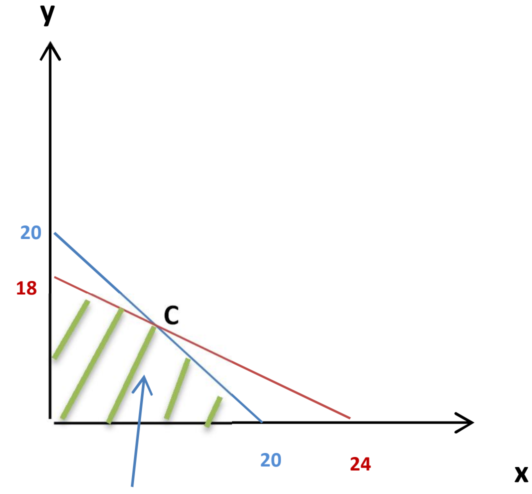
x- intercept (set y = 0) is

3 x + 0 = 72 , then x = 24 ,the point is (24, 0)

y- intercept (set x = 0) is

0 + 4 y = 72 , then y = 18 ,the point is (0, 18)

and x = 0, y- axis y = 0, x -axis.



feasible region

To find the intersection point C

We need to solve the system of equations

$$x + y = 20 \quad \ldots\ldots (1)$$

$$3x + 4y = 72 \quad \ldots\ldots\ldots (2)$$

Multiply equation (1) by -3 and added to (2) to get

y = 12 and x = 8 then the point C is (8, 12)

Now we substitute the corner points into the objective function Z then fined the maximum one

Corner points	Value of Z
(20,0)	80
(0, 18)	90
(8, 12)	92

The maximum value of the objective function

Finally, the optimal solutions x = 8 and y = 12 which is unique

Example: Multiple Optimal Solutions

Solve the following LPP by graphical method

Max $\quad Z = 100\ x_1 + 40\ x_2$

Subject to

$5\ x_1 + 2\ x_2 \leq 1000$

$3\ x_1 + 2\ x_2 \leq 900$

$x_1 + 2\ x_2 \leq 500$

$x_1, x_2 \geq 0$

SOLUTION

First, we need to draw the straight lines of constrains. This means finding x_1- intercept and x_2 - intercept.

For the constrain $5\ x_1 + 2\ x_2 \leq 1000$

x_1 - intercept (set $x_2 = 0$) is

$5\ x_1 + 0 = 1000$, then $x_1 = 200$, the point (200, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + 2\ x_2 = 1000$, then $x_2 = 500$, the point is (0, 500)

For the constrain $3\ x_1 + 2\ x_2 \leq 900$

x_1 - intercept (set $x_2 = 0$) is

$3\ x_1 + 0 = 900$, then $x_1 = 300$, the point (300, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + 2\ x_2 = 900$, then $x_2 = 450$, the point is (0, 450)

For the constrain $x_1 + 2\ x_2 \leq 500$

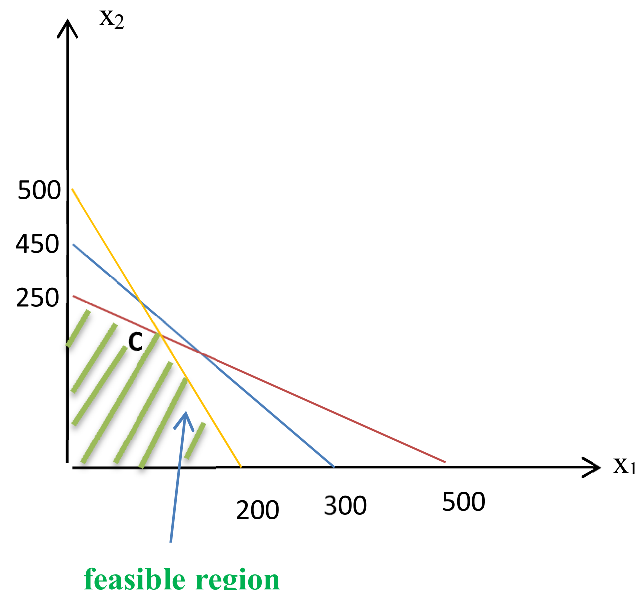
x_1 - intercept (set $x_2 = 0$) is

$x_1 + 0 = 500$, then $x_1 = 500$, the point (500, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + 2\ x_2 = 500$, then $x_2 = 250$, the point is (0, 250)

and $x_1 = 0$, x_2 axis $x_2 = 0$, x_1 axis.



To find the intersection point C

We need to solve the system of equations

$$x_1 + 2\,x_2 = 500 \qquad \ldots\ldots. (1)$$

$$5\,x_1 + 2\,x_2 = 1000 \qquad \ldots\ldots\ldots(2)$$

Multiply equation (1) by -1 and added to (2) to get

$x_1 = 125$ and $x_2 = 187.5$ then the point C = (125, 187.5)

Now we substitute the corner points into the objective function Z then fined the maximum one

Corner points	Value of Z
(200,0)	20000
(0, 250)	10000
(125, 187.5)	20000

The maximum values of the objective function Z

Finally, there are multiple optimum solutions for the LPP. (0, 250) and (125, 187.5)

Example: Unbounded Solutions

Use graphical method to solve the following LPP.

Max $Z = 3x_1 + 2x_2$

Subject to

$5x_1 + x_2 \geq 10$

$x_1 + x_2 \geq 6$

$x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

SOLUTION

First, we need to draw the straight lines of constrains. This means finding x_1- intercept and x_2 - intercept.

For the constrain $5x_1 + x_2 \geq 10$

x_1 - intercept (set $x_2 = 0$) is

$5x_1 + 0 = 10$, then $x_1 = 2$, the point (2, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + x_2 = 10$, then $x_2 = 10$, the point is (0, 10)

For the constrain $x_1 + x_2 \geq 6$

x_1 - intercept (set $x_2 = 0$) is

$x_1 + 0 = 6$, then $x_1 = 6$, the point (6, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + x_2 = 6$, then $x_2 = 6$, the point is (0, 6)

For the constrain $x_1 + 4x_2 \geq 12$

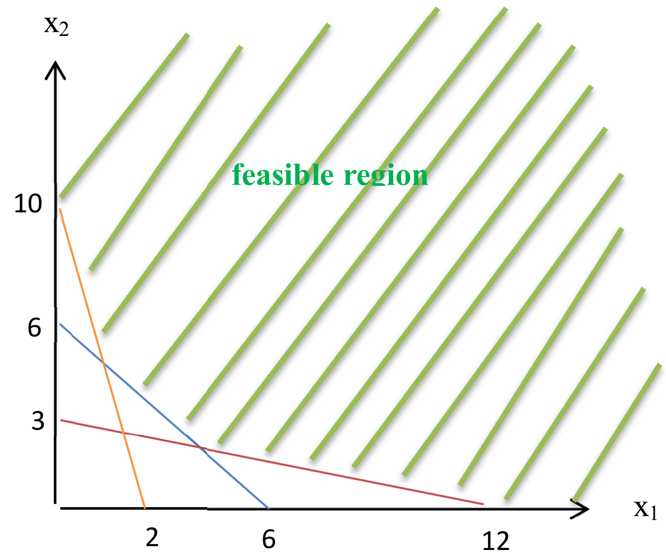
x_1 - intercept (set $x_2 = 0$) is

$x_1 + 0 = 12$, then $x_1 = 12$, the point (12, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + 4x_2 = 12$, then $x_2 = 3$, the point is (0, 3)

and $x_1 = 0$, x_2 axis $x_2 = 0$, x_1 axis.



The feasible region is unbounded. Thus, the maximum value of Z occurs at infinity; hence, the problem has an unbounded solution.

Example: No Feasible Solution

Use graphical method to solve the following LPP.

Max $Z = x_1 + x_2$

Subject to

$x_1 + x_2 \leq 1$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

<u>SOLUTION</u>

First, we need to draw the straight lines of constrains. This means finding x_1- intercept and x_2 - intercept.

For the constrain $x_1 + x_2 \leq 1$

x_1 - intercept (set $x_2 = 0$) is

$x_1 + 0 = 1$, then $x_1 = 1$, the point (1, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + x_2 = 1$, then $x_2 = 1$, the point is (0, 1)

For the constrain $x_1 + x_2 \geq 3$

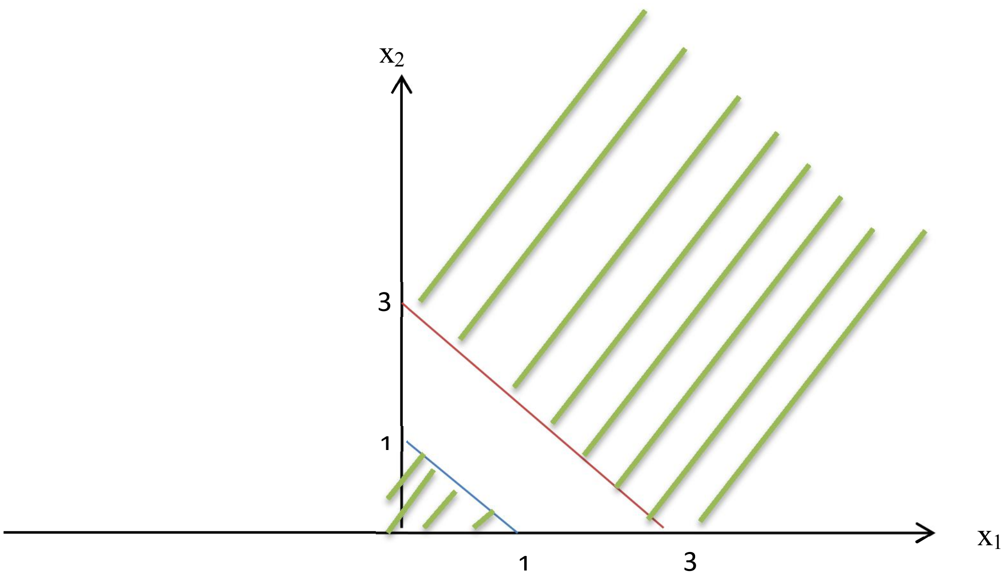
x_1 - intercept (set $x_2 = 0$) is

$x_1 + 0 = 3$, then $x_1 = 3$, the point (3, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + x_2 = 3$, then $x_2 = 3$, the point is (0, 3)

and $x_1 = 0$, x_2 axis $x_2 = 0$, x_1 axis.



We cannot find a feasible region for this problem. So the problem can not be solved, hence, the problem has no solution.

Now, let's find the solutions of previous examples graphically

Example: Giapetto's Woodcarving

Giapetto's Woodcarving, manufactures two types of wooden toys: soldiers and trains. A soldier sells for $27 and uses $10 of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14. A train sells for $21 and uses $9 of raw materials. Each train built increases Giapetto's variable labor and overhead costs by $10. The soldiers and trains require two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Giapetto wants to maximize weekly profit (revenues _ costs). How many of each toy should be made each week to maximize profits? Formulate the Mathematical LP Model of the Problem.

SOLUTION

Maximize $Z = 3x_1 + 2x_2$

Subject to

$$2x_1 + x_2 \leq 100$$

$$x_1 + x_2 \leq 80$$

$$x_1 \leq 40$$

$$x_1, x_2 \geq 0$$

For the constrain $2x_1 + x_2 \leq 100$

x_1 - intercept (set $x_2 = 0$) is

$2x_1 + 0 = 100$, then $x_1 = 1$, the point (50, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + x_2 = 100$, then $x_2 = 100$, the point is (0, 100)

For the constrain $x_1 + x_2 \leq 80$

x_1 - intercept (set $x_2 = 0$) is

$x_1 + 0 = 80$, then $x_1 = 1$, the point (80, 0)

x_2 - intercept (set $x_1 = 0$) is

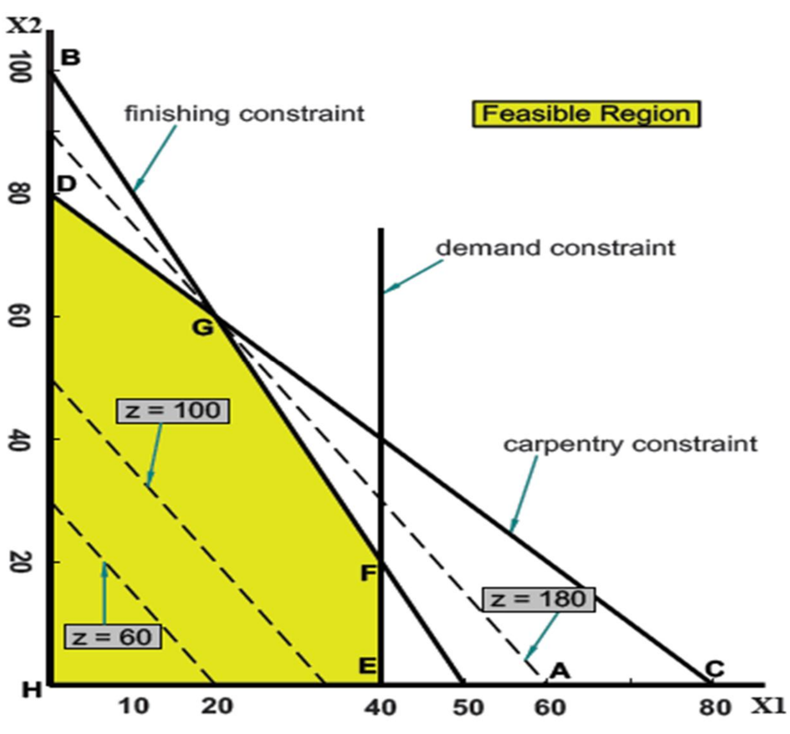
$0 + x_2 = 80$, then $x_2 = 80$, the point is (0, 80)

For the constrain $x_1 \leq 40$

$x_1 = 40$ (straight line perpendicular on x_1 – axis)

and $x_1 = 0$, x_2 axis $x_2 = 0$, x_1 axis.

Now, plot the lines to find the feasible region



To find the intersection point G

We need to solve the system of equations

$$2x_1 + x_2 = 100 \quad \ldots\ldots. (1)$$

$$x_1 + x_2 = 80 \quad \ldots\ldots\ldots(2)$$

Multiply equation (2) by -1 and added to (2) to get

$x_1 = 20$ and $x_2 = 60$ then the point G $= (20, 60)$

To find the intersection point F

We need to solve the system of equations

$$2x_1 + x_2 = 100 \quad \ldots\ldots. (1)$$

$$x_1 = 40 \quad \ldots\ldots\ldots(2)$$

$x_1 = 40$ and $x_2 = 20$ then the point G $= (40, 20)$

Now we substitute the corner points into the objective function Z then fined the maximum one

Corner points	Value of Z
(0,80)	160
(40, 0)	120
(20, 60)	180
(40, 20)	160

The maximum value of the objective function Z

Finally, the optimal solutions (20, 60) which is unique

Example: Advertisement Example

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

SOLUTION

The decision variables are

x_1 = the number of comedy spots

x_2 = the number of football spots

The model of the problem:

Min $z = 50x_1 + 100x_2$

Subject to

$7\, x_1 + 2\, x_2 \geq 28$ (high income women)

$2\, x_1 + 12\, x_2 \geq 24$ (high income men)

$x_1, x_2 \geq 0$

For the constrain $7\, x_1 + 2\, x_2 \geq 28$

x_1 - intercept (set $x_2 = 0$) is

$7x_1 + 0 = 28$, then $x_1 = 1$, the point (4, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + 2\, x_2 = 28$, then $x_2 = 100$, the point is (0, 14)

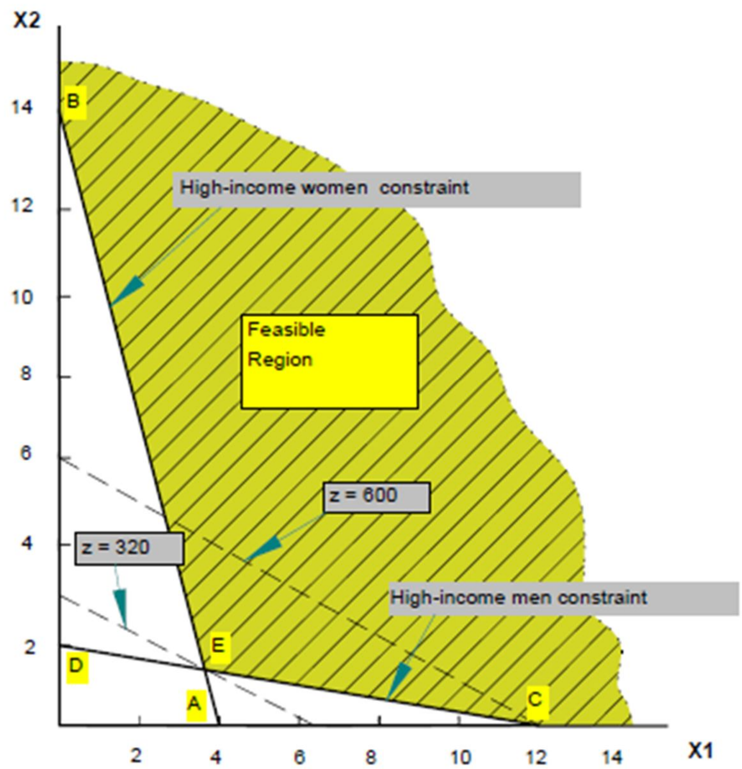
For the constrain $2\, x_1 + 12\, x_2 \geq 24$

x_1 - intercept (set $x_2 = 0$) is

$2x_1 + 0 = 24$, then $x_1 = 12$, the point (12, 0)

x_2 - intercept (set $x_1 = 0$) is

$0 + 12 x_2 = 24$, then $x_2 = 2$, the point is (0, 2)



To find the intersection point E

We need to solve the system of equations

$$7 x_1 + 2 x_2 = 28 \quad \ldots\ldots. (1)$$

$$2 x_1 + 12 x_2 = 24 \quad \ldots\ldots\ldots(2)$$

Multiply equation (1) by -6 and added to (2) to get

$x_1 = 3.6$ and $x_2 = 1.4$ then the point E = (3.6, 1.4)

Now we substitute the corner points into the objective function Z then fined the maximum one

Corner points	Value of Z
(0,14)	1400
(3.6, 1.4)	320
(12, 0)	600

The minimum value of the objective function Z

Finally, the optimal solution is (3.6, 1.4) which is unique

THE SIMPLEX METHOD

If the linear programming problem has number of variables greater than two , the suitable and the most widely used method is Simplex Method. The simplex method is an iterative process, through which it reaches ultimately to the minimum or maximum value of the objective function. This method is used to solve even very large LPs. In many industrial applications, the simplex algorithm is used to solve LPs with thousands of constraints and variables.

Standard Form of LPP

We have to convert the LPP into the standard form of LPP before the use of simplex method. The standard form of the LPP should have the following characteristics;

i. All the constraints should be expressed as equations by adding slack or surplus and / or artificial variables.

ii. The right hand side of each constraint should be made non negative if it is not, this should be done by multiplying both sides of the resulting constraints by -1.

iii. The objective function should be of the maximization type

The general standard form of the LPP is expressed as follows;