Corner points	Value of Z
(0,14)	1400
(3.6, 1.4)	320
(12, 0)	600

The minimum value of the objective function Z

Finally, the optimal solution is (3.6, 1.4) which is unique

THE SIMPLEX METHOD

If the linear programming problem has number of variables greater than two , the suitable and the most widely used method is Simplex Method. The simplex method is an iterative process, through which it reaches ultimately to the minimum or maximum value of the objective function. This method is used to solve even very large LPs. In many industrial applications, the simplex algorithm is used to solve LPs with thousands of constraints and variables.

Standard Form of LPP

We have to convert the LPP into the standard form of LPP before the use of simplex method. The standard form of the LPP should have the following characteristics;

i. All the constraints should be expressed as equations by adding slack or surplus and / or artificial variables.
ii. The right hand side of each constraint should be made non negative if it is not, this should be done by multiplying both sides of the resulting constraints by -1.
iii. The objective function should be of the maximization type

The general standard form of the LPP is expressed as follows;

$$Optimize\ \ Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0S_1 + 0S_2 + \ldots + 0S_m$$

subjected to the constraints

$$\begin{array}{llllll}
a_{11}x_1 & +a_{12}x_2 & +\ldots & +a_{1j}x_j & +\ldots a_{1n}x_n + S_1(\leq = \geq)b_1 \\
a_{21}x_1 & +a_{22}x_2 & +\ldots & +a_{2j}x_j & +\ldots a_{2n}x_n + S_2(\leq = \geq)b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{i1}x_1 & +a_{i2}x_2 & +\ldots & +a_{ij}x_j & +\ldots a_{in}x_n + S_n(\leq = \geq)b_i \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1}x_1 & +a_{m2}x_2 & +\ldots & +a_{mi}x_j & +\ldots a_{mn}x_n + S_m(\leq = \geq)b_n
\end{array}$$

and non negativity constraints

$$x_1, x_2, \ldots, x_n, S_1, S_2, \ldots, S_m \geq 0$$

Where (S_1 , …, S_m) are the added variables

Remark:

- ***Slack variables*** are defined as the non-negative variables which are added in the LHS of the constraints to convert the inequality ($\leq$) into an equation. Such variables are added to the original objective function with zero coefficients.
- ***Surplus variables*** are removed from the LHS of the constraints to convert the inequality ($\geq$) into an equation. Surplus variables, like slack variable carry a zero coefficient in the objective function. These variables are also called negative slack variables.
- ***Artificial variables*** are also defined as the non-negative variables which are added in the LHS of the constraints to convert equality into the standard form of simplex.
- **Basic solution (BS) :** For a system of m simultaneous linear equations in n variables (n > m), a solution obtained by setting (n-m) variables equal to zero and solving for the remaining+ variables is called a basic solution. Such m variables (of course, some of them may be zero) are called **basic variables (active variables)** and remaining (n-m) zero-valued variables are called **non-basic variables (non-active variable)**.

  Basic feasible solution (BFS) A basic feasible solution is a basic solution which also satisfies the non-negativity restrictions, that is all basic variables are non-negative. Basic solutions are of two types

a) **Non-degenerate BFS**: A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_j (j = 1, 2,..., m). In other words, all basic m variables are positive, and the remaining (n -m) variables will be all zero.

b) **Degenerate BFS**: A basic feasible solution is called degenerate, if one or more basic variables are zero-valued.

Simplex Method for Standard Maximization Problem

1) Convert the system of inequality to equation (constraints) to equations using slack variables.
2) Set the objective function equal to zero for standard maximization problems.
3) Create a simplex table or tableau and label the active or basic variables.
4) Select the pivot column. This is the column with the most negative number on the left side of the bottom row. **(If all numbers on the bottom row are nonnegative, we are done.)**
5) Select the pivot row. Divide each entry in the constant column by the corresponding positive entry in the pivot column. The smallest positive ratio indicates the pivot row.
6) Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.
7) Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8) Repeat the process by identifying the most negative entry on the left side of the last row.
9) Once the left side of last row is all nonnegative, the solution can be found. The value of each row variables or active variables is equal to the right most entry in the row. All inactive variables equal zero.

Example:

Maximize $Z = 2x + 5y$

Subject to

$$2x + y \le 5$$

$$x + 2y \le 4$$

$$x, y \ge 0$$

Convert to a system of equations and set the objective function equal 0

$$2x + y + s = 5$$

$$x + 2y + t = 4$$

$$-2x - 5y + Z = 0$$

This system can be written as follows

$$2x + y + s + 0t + 0Z = 5$$

$$x + 2y + 0s + t + 0Z = 4$$

$$-2x - 5y + 0s + 0t + Z = 0$$

	x	y	s	t	Z	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
Z	-2	-5	0	0	1	0

S, t , Z are the active variables that represent the identity columns in the table

The initial basic feasible solutions are the active variables s = 5 , t = 4, Z = 0

From the last row the most negative number is -5

	x	y	s	t	Z	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
Z	-2	-5	0	0	1	0

pivot column

Now, to find the pivot row we need to

Divide the last entry of the 1st row by 1 (1st number in the pivot column) then we get the ratio ($\frac{5}{1}$) and the last entry of the 2nd row by 2 (2nd number in the pivot column) then we get the ratio ($\frac{4}{2} = 2$)

Since 2 is the smallest ratio , we will choose the corresponding row as the pivot row

	x	y	s	t	Z	
s	2	1	1	0	0	5
t	1	$\boxed{2}$	0	1	0	4
Z	-2	-5	0	0	1	0

Pivot row

Pivot

Convert the pivot to be 1 by dividing it and the pivot row by 2

$R_2 = R_2 / 2$

	x	y	s	t	Z	
s	2	1	1	0	0	5
t	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
Z	-2	-5	0	0	1	0

Change the pivot column into identity column

$R_1 = R_1 - R_2$

$R_3 = R_3 + 5R_2$

	x	y	s	t	Z	
s	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	3
t	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
Z	$\frac{1}{2}$	0	0	$\frac{5}{2}$	1	10

The row t doesn't contain 1 or zeros , so it is no longer active variable and we will change t in the row by y (becomes the active variable)

	x	y	s	t	Z	
s	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	3
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
Z	$\frac{1}{2}$	0	0	$\frac{5}{2}$	1	10

Since all the entries of the last row are 0 and positive , this mean we have to stop

Then , the active variables are y = 2 , s = 3 , and Z = 10 . Then the non- active variables are equal to zero (hence x = 0) .

Finally, the maximum of Z is 10 at (0, 2)