

The Two-Phase Simplex Method

When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method. In the two-phase simplex method, we add artificial variables to the same constraints as we did in the Big M method. Then we find a bfs to the original LP by solving the Phase I LP. In the Phase I LP, the objective function is to minimize the sum of all artificial variables. At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.

The following steps describe the two-phase simplex method. Note that steps 1–3 for the two-phase simplex are identical to steps 1–4 for the Big M method.

Steps

- 1) Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1 .
- 2) Identify each constraint that is now (after step 1) \geq or $=$ constraint. In step 3, we will add an artificial variable to each constraint.
- 3) Convert each inequality constraint to the standard form. If constraint i is \leq constraint, then add a slack variable s_i . If constraint i is \geq constraint, subtract an excess variable e_i .
- 4) If (after step 2) constraint i is \geq or $=$ constraint, add an artificial variable a_i .
- 5) For now, ignore the original LP's objective function. Instead solve an LP whose objective function is $\min W = (\text{sum of all the artificial variables})$. This is called the **Phase I LP**. The act of solving the Phase I LP will force the artificial variables to be zero.

Because each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases:

Case 1 The optimal value of W is greater than zero. In this case, the original LP has no feasible solution.

Case 2 The optimal value of W is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, we drop all columns in the optimal Phase I tableau that corresponds to the artificial variables. We

now combine the original objective function with the constraints from the optimal Phase I tableau. This yields the **Phase II LP**. The optimal solution to the Phase II LP is the optimal solution to the original LP.

Case 3 The optimal value of W is equal to zero and at least one artificial variable is in the optimal Phase I basis. In this case, we can find the optimal solution to the original LP if at the end of Phase I we drop from the optimal Phase I tableau all nonbasic artificial variables and any variable from the original problem that has a negative coefficient in objective row of the optimal Phase I tableau.

Before solving examples illustrating Cases 1–3, we briefly discuss why $W > 0$ corresponds to the original LP having no feasible solution and $W = 0$ corresponds to the original LP having at least one feasible solution.

Example: Case 2

Maximize $Z = 4x + 5y$

Subject to

$$2x + 3y \leq 6$$

$$3x + y \geq 3$$

$$x, y \geq 0$$

First, convert the constraints to equations

Maximize $Z = 4x + 5y$

Subject to

$$2x + 3y + s_1 = 6$$

$$3x + y - e + a = 3$$

$$x, y, s_1, e, a \geq 0$$

Second, ignore the original LP's objective function. Instead solve an LP whose objective function is

$$\text{Minimize } W = a$$

Then, **Phase I LP** will be as follows

$$\text{Minimize } W = a$$

Subject to

$$2x + 3y + s + 0e + 0a + 0W = 6$$

$$3x + y + 0s - e + a + 0W = 3$$

$$x, y, s, e, a \geq 0$$

After setting the objective function equal to zero we have the following tableau

	x	y	s	e	a	W	
x	2	3	1	0	0	0	6
y	3	1	0	-1	1	0	3
W	0	0	0	0	-1	1	0

$$R_3 = R_3 + R_2$$

Now, eliminate the artificial variable in the objective function row by converting it to 0

	x	y	s	e	a	W	
s	2	3	1	0	0	0	6
a	3	1	0	-1	1	0	3
W	3	1	0	-1	0	1	3



Since here we always have minimization for the objective function, we will choose the greatest positive number in the objective row. by this way we will choose the pivot column and by the ratio test we choose the pivot row.

	x	y	s	e	a	W	
s	0	7/3	1	2/3	-2/3	0	4
x	1	1/3	0	-1/3	1/3	0	1
W	0	0	0	0	-1	1	0

Optimal basis

Since there is no positive number in the objective row, we will stop. The optimal basis $s = 4$ and $x = 1$. The optimal objective value $W = 0$ and a is not in the optimal Phase I basis. Therefore, we have **Case 2**

Hence, we can generate **Phase II** initial tableau by changing the objective function row into the original objective function ($Z = 4x + 5y$). Also, drop the artificial column

Phase II

$R_3 = R_3 + 4 R_2$

	x	y	s	e	Z	
s	0	7/3	1	2/3	0	4
x	1	1/3	0	-1/3	0	1
Z	-4	-5	0	0	1	0

Since x is one of the basis we need to convert -4 in objective row to 0

	x	y	s	e	Z	
s	0	7/3	1	2/3	0	4
x	1	1/3	0	-1/3	0	1
Z	0	-11/3	0	-4/3	1	4

Use Simplex Method . Choose the most negative element in the objective row

	x	y	s	e	Z	
y	0	1	3/7	2/7	0	12/7
x	1	0	-1/7	-3/7	0	3/7
Z	0	0	11/7	-2/7	1	72/7

	x	y	s	e	Z	
e	0	7/2	3/2	1	0	6
x	1	3/2	1/2	0	0	3
Z	0	1	2	0	1	12

Since all the elements of the objective row are nonnegative, then we will stop. The optimal solution to the Phase II LP is the optimal solution to the original LP and equal to $x = 3$, $y = 0$ with $Z = 12$

Example: Case 1

Minimize $Z = 2x + 3y$

Subject to

$$\frac{1}{2}x + \frac{1}{4}y \leq 4$$

$$x + 3y \geq 36$$

$$x + y = 10$$

$$x, y \geq 0$$

Phase I LP

Minimize $W = a_1 + a_2$

Subject to

$$\frac{1}{2}x + \frac{1}{4}y + s + 0e + 0a_1 + 0a_2 + 0W = 4$$

$$x + 3y + 0s - e + a_1 + 0a_2 + 0W = 36$$

$$x + y + 0s + 0e + 0a_1 + a_2 + 0W = 10$$

$$x, y, s, e, a_1, a_2 \geq 0$$

After setting the objective function equal to zero we have the following tableau

	x	y	s	e	a ₁	a ₂	W	
s	1/2	1/4	1	0	0	0	0	4
	1	3	0	-1	1	0	0	36
	1	1	0	0	0	1	0	10
W	0	0	0	0	-1	-1	1	0

$R_4 = R_4 + R_2$

Now, eliminate the artificial variables in the objective function row by converting it to 0

	x	y	s	e	a ₁	a ₂	W	
s	1/2	1/4	1	0	0	0	0	4
a ₁	1	3	0	-1	1	0	0	36
	1	1	0	0	0	1	0	10
W	1	3	0	-1	0	-1	1	36

$$R_4 = R_4 + R_3$$

	x	y	s	e	a ₁	a ₂	W	
s	1/2	1/4	1	0	0	0	0	4
a ₁	1	3	0	-1	1	0	0	36
a ₂	1	1	0	0	0	1	0	10
W	2	4	0	-1	0	0	1	46

$$R_4 = R_4 - 4 R_3$$

Choose the greatest positive number in the objective row . By this way we choose the pivot column and by the ratio test we choose the pivot row .

	x	y	s	e	a ₁	a ₂	W	
s	1/4	0	1	0	0	-1/4	0	3/2
a ₁	-2	0	0	-1	1	-3	0	6
y	1	1	0	0	0	1	0	10
W	-2	0	0	-1	0	-4	1	6

Since there is no positive number in the objective row, we will stop. The optimal basis $s = 3/2$, $a_1 = 6$, and $y = 10$. The optimal objective value is $W = 6 > 0$. then we have **Case 2** which means the original LP has no feasible solution.

Example: Case 3

Maximize $Z = 40 x_1 + 10 x_2 + 7 x_5 + 14 x_6$

Subject to

$$x_1 - x_2 + 2 x_5 = 0$$

$$-2 x_1 + x_2 - 2 x_5 = 0$$

$$x_1 + x_3 + x_5 - x_6 = 3$$

$$x_2 + x_3 + x_4 + 2 x_5 + x_6 = 4$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Convert the constrains to equations by adding artificial variables (active variables).

Note : we can use x_4 as a basic (active) variable . Therefor we will add only 3 artificial variables for the first three variables.

Phase I LP

Minimize $W = a_1 + a_2 + a_3$

Subject to

$$x_1 - x_2 + 2 x_5 + a_1 + 0 a_2 + 0 a_3 + 0W = 0$$

$$-2 x_1 + x_2 - 2 x_5 + 0 a_1 + a_2 + 0 a_3 + 0W = 0$$

$$x_1 + x_3 + x_5 - x_6 + 0 a_1 + 0 a_2 + a_3 + 0W = 3$$

$$x_2 + x_3 + x_4 + 2 x_5 + x_6 + 0 a_1 + 0 a_2 + 0a_3 + 0W = 4$$

$$x_1, x_2, x_3, x_4, x_5, x_6, a_1, a_2, a_3 \geq 0$$

	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	W	
a_1	1	-1	0	0	2	0	1	0	0	0	0
a_2	-2	1	0	0	-2	0	0	1	0	0	0
a_3	1	0	1	0	1	-1	0	0	1	0	3
x_4	0	1	1	1	2	1	0	0	0	0	4
W	0	0	0	0	0	0	-1	-1	-1	1	0

$$R_5 = R_5 + R_1$$

	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	a ₁	a ₂	a ₃	W	
a ₁	1	-1	0	0	2	0	1	0	0	0	0
a ₂	-2	1	0	0	-2	0	0	1	0	0	0
a ₃	1	0	1	0	1	-1	0	0	1	0	3
x ₄	0	1	1	1	2	1	0	0	0	0	4
W	1	-1	0	0	2	0	0	-1	-1	1	0

$$R_5 = R_5 + R_2$$

	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	a ₁	a ₂	a ₃	W	
a ₁	1	-1	0	0	2	0	1	0	0	0	0
a ₂	-2	1	0	0	-2	0	0	1	0	0	0
a ₃	1	0	1	0	1	-1	0	0	1	0	3
x ₄	0	1	1	1	2	1	0	0	0	0	4
W	-1	0	0	0	0	0	0	0	-1	1	0

$$R_5 = R_5 + R_3$$

	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	a ₁	a ₂	a ₃	W	
a ₁	1	-1	0	0	2	0	1	0	0	0	0
a ₂	-2	1	0	0	-2	0	0	1	0	0	0
a ₃	1	0	1	0	1	-1	0	0	1	0	3
x ₄	0	1	1	1	2	1	0	0	0	0	4
W	0	0	1	0	1	-1	0	0	0	1	3

$$R_4 = R_4 - R_3$$

$$R_5 = R_5 - R_3$$

Now we will start simplex method. Choose the greatest positive number in the objective row

	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	a ₁	a ₂	a ₃	W	
a ₁	1	-1	0	0	2	0	1	0	0	0	0
a ₂	-2	1	0	0	-2	0	0	1	0	0	0
x ₃	1	0	1	0	1	-1	0	0	1	0	3
x ₄	-1	2	0	1	1	2	0	0	-1	0	1
W	-1	0	0	0	0	0	0	0	-1	1	0

Since there is no positive number in the objective row, we will stop. The optimal basis are $a_1 = 0$, $a_2 = 0$, $x_3=3$ and $x_4 =1$. The optimal objective value $W = 0$ and we have two artificial variables a_1 and a_2 in the optimal Phase I basis. Therefore, we have **Case 3**.

Hence, we can generate **Phase II** initial tableau by changing the objective function row into the original objective function. Also, drop the columns of the nonbasic (not active) artificial variable(s), which is a_3 in this example, and of the original variables with a negative coefficient in objective row of the optimal Phase-I tableau. There is only one such variable, which is x_1

Phase II

$R_4 = R_4 / 2$

	x_2	x_3	x_4	x_5	x_6	a_1	a_2	Z	
a_1	-1	0	0	2	0	1	0	0	0
a_2	1	0	0	-2	0	0	1	0	0
x_3	0	1	0	1	-1	0	0	0	3
x_4	2	0	1	1	2	0	0	0	1
Z	-10	0	0	-7	-14	0	0	1	0

Choose the most negative number in the objective row

$R_3 = R_3 + R_4$

$R_5 = R_5 + 14 R_4$

	x_2	x_3	x_4	x_5	x_6	a_1	a_2	Z	
a_1	-1	0	0	2	0	1	0	0	0
a_2	1	0	0	-2	0	0	1	0	0
x_3	0	1	0	1	-1	0	0	0	3
x_4	1	0	1/2	1/2	1	0	0	0	1/2
Z	-10	0	0	-7	-14	0	0	1	0

	x_2	x_3	x_4	x_5	x_6	a_1	a_2	Z	
a_1	-1	0	0	2	0	1	0	0	0
a_2	1	0	0	-2	0	0	1	0	0
x_3	1	1	1/2	3/2	0	0	0	0	7/2
x_6	1	0	1/2	1/2	1	0	0	0	1/2
Z	4	0	7	0	0	0	0	1	7

Since all the elements of the objective row are nonnegative, then we will stop. The optimal solution to the Phase II LP is the optimal solution to the original LP and equal to $x_1 = 0$, $x_2 = 0$, $x_3 = 7/2$, $x_4 = 0$, $x_5 = 0$, and $x_6 = 1/2$ with $Z = 7$