It is important to understand the main ideas in the proof above, because we will follow these ideas to find power series solutions to differential equations. So we now summarize the main steps in the proof above:

(a) Write a power series expansion of the solution centered at a regular point x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

(b) Introduce the power series expansion above into the differential equation and find a *recurrence relation* among the coefficients a_n.

(c) Solve the recurrence relation in terms of free coefficients.

(d) If possible, add up the resulting power series for the solutions y_1, y_2.

We follow these steps in the examples below to find solutions to several differential equations. We start with a first order constant coefficient equation, and then we continue with a second order constant coefficient equation. The last two examples consider variable coefficient equations.

Example 3.1.2. Find a power series solution y around the point $x_0 = 0$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: We already know every solution to this equation. This is a first order, linear, differential equation, so using the method of integrating factor we find that the solution is

$$y(x) = a_0\, e^{-c\,x}, \qquad a_0 \in \mathbb{R}.$$

We are now interested in obtaining such solution with the power series method. Although this is not a second order equation, the power series method still works in this example. Propose a solution of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)}.$$

We can start the sum in y' at $n = 0$ or $n = 1$. We choose $n = 1$, since it is more convenient later on. Introduce the expressions above into the differential equation,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} + c \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Relabel the first sum above so that the functions x^{n-1} and x^n in the first and second sum have the same label. One way is the following,

$$\sum_{n=0}^{\infty} (n+1) a_{(n+1)}\, x^n + \sum_{n=0}^{\infty} c\, a_n\, x^n = 0$$

We can now write down both sums into one single sum,

$$\sum_{n=0}^{\infty} \big[(n+1) a_{(n+1)} + c\, a_n\big]\, x^n = 0.$$

Since the function on the left-hand side must be zero for every $x \in \mathbb{R}$, we conclude that every coefficient that multiplies x^n must vanish, that is,

$$(n+1) a_{(n+1)} + c\, a_n = 0, \qquad n \geqslant 0.$$

The last equation is called a *recurrence relation* among the coefficients a_n. The solution of this relation can be found by writing down the first few cases and then guessing the general expression for the solution, that is,

$$n = 0, \qquad a_1 = -c\, a_0 \qquad \Rightarrow \qquad a_1 = -c\, a_0,$$

$$n = 1, \qquad 2a_2 = -c\, a_1 \qquad \Rightarrow \qquad a_2 = \frac{c^2}{2!}\, a_0,$$

$$n = 2, \qquad 3a_3 = -c\, a_2 \qquad \Rightarrow \qquad a_3 = -\frac{c^3}{3!}\, a_0,$$

$$n = 3, \qquad 4a_4 = -c\, a_3 \qquad \Rightarrow \qquad a_4 = \frac{c^4}{4!}\, a_0.$$

One can check that the coefficient a_n can be written as

$$a_n = (-1)^n \frac{c^n}{n!}\, a_0,$$

which implies that the solution of the differential equation is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{c^n}{n!}\, x^n \quad \Rightarrow \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c\,x)^n}{n!} \quad \Rightarrow \quad y(x) = a_0\, e^{-c\,x}.$$

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Example 3.1.3. Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: We know that the solution can be found computing the roots of the characteristic polynomial $r^2 + 1 = 0$, which gives us the solutions

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

We now recover this solution using the power series,

$$y = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)}, \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n\, x^{(n-2)}.$$

Introduce the expressions above into the differential equation, which involves only the function and its second derivative,

$$\sum_{n=2}^{\infty} n(n-1) a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Relabel the first sum above, so that both sums have the same factor x^n. One way is,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)}\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Now we can write both sums using one single sum as follows,

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1) a_{(n+2)} + a_n\big]\, x^n = 0 \quad \Rightarrow \quad (n+2)(n+1) a_{(n+2)} + a_n = 0. \quad n \geqslant 0.$$

The last equation is the *recurrence relation*. The solution of this relation can again be found by writing down the first few cases, and we start with even values of n, that is,

$$n = 0, \qquad (2)(1)a_2 = -a_0 \qquad \Rightarrow \qquad a_2 = -\frac{1}{2!}\,a_0,$$

$$n = 2, \qquad (4)(3)a_4 = -a_2 \qquad \Rightarrow \qquad a_4 = \frac{1}{4!}\,a_0,$$

$$n = 4, \qquad (6)(5)a_6 = -a_4 \qquad \Rightarrow \qquad a_6 = -\frac{1}{6!}\,a_0.$$

One can check that the even coefficients a_{2k} can be written as

$$a_{2k} = \frac{(-1)^k}{(2k)!}\,a_0.$$

The coefficients a_n for the odd values of n can be found in the same way, that is,

$$n = 1, \qquad (3)(2)a_3 = -a_1 \qquad \Rightarrow \qquad a_3 = -\frac{1}{3!}\,a_1,$$

$$n = 3, \qquad (5)(4)a_5 = -a_3 \qquad \Rightarrow \qquad a_5 = \frac{1}{5!}\,a_1,$$

$$n = 5, \qquad (7)(6)a_7 = -a_5 \qquad \Rightarrow \qquad a_7 = -\frac{1}{7!}\,a_1.$$

One can check that the odd coefficients a_{2k+1} can be written as

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\,a_1.$$

Split the sum in the expression for y into even and odd sums. We have the expression for the even and odd coefficients. Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\,x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\,x^{2k+1}.$$

One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

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Example 3.1.4. Find the first four terms of the power series expansion around the point $x_0 = 1$ of each fundamental solution to the differential equation

$$y'' - x\,y' - y = 0.$$

Solution: This is a differential equation we cannot solve with the methods of previous sections. This is a second order, variable coefficients equation. We use the power series method, so we look for solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

We start working in the middle term in the differential equation. Since the power series is centered at $x_0 = 1$, it is convenient to re-write this term as $x\,y' = [(x-1)+1]\,y'$, that is,

$$x\,y' = \sum_{n=1}^{\infty} n a_n x (x-1)^{n-1}$$
$$= \sum_{n=1}^{\infty} n a_n \big[(x-1)+1\big](x-1)^{n-1}$$
$$= \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}. \qquad (3.1.4)$$

As usual by now, the first sum on the right-hand side of Eq. (3.1.4) can start at $n = 0$, since we are only adding a zero term to the sum, that is,

$$\sum_{n=1}^{\infty} n a_n (x-1)^n = \sum_{n=0}^{\infty} n a_n (x-1)^n;$$

while it is convenient to relabel the second sum in Eq. (3.1.4) follows,

$$\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{(n+1)} (x-1)^n;$$

so both sums in Eq. (3.1.4) have the same factors $(x-1)^n$. We obtain the expression

$$x\,y' = \sum_{n=0}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{(n+1)} (x-1)^n$$
$$= \sum_{n=0}^{\infty} \big[n a_n + (n+1) a_{(n+1)}\big](x-1)^n. \qquad (3.1.5)$$

In a similar way relabel the index in the expression for y'', so we obtain

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-1)^n. \qquad (3.1.6)$$

If we use Eqs. (3.1.5)-(3.1.6) in the differential equation, together with the expression for y, the differential equation can be written as follows

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-1)^n - \sum_{n=0}^{\infty} \big[n a_n + (n+1) a_{(n+1)}\big](x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0.$$

We can now put all the terms above into a single sum,

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1) a_{(n+2)} - (n+1) a_{(n+1)} - n a_n - a_n\Big] (x-1)^n = 0.$$

This expression provides the *recurrence relation* for the coefficients a_n with $n \geqslant 0$, that is,

$$(n+2)(n+1) a_{(n+2)} - (n+1) a_{(n+1)} - (n+1) a_n = 0$$
$$(n+1)\Big[(n+2) a_{(n+2)} - a_{(n+1)} - a_n\Big] = 0,$$

which can be rewritten as follows,

$$(n+2) a_{(n+2)} - a_{(n+1)} - a_n = 0. \qquad (3.1.7)$$

We can solve this recurrence relation for the first four coefficients,

$$n = 0 \qquad 2a_2 - a_1 - a_0 = 0 \qquad \Rightarrow \qquad a_2 = \frac{a_1}{2} + \frac{a_0}{2},$$
$$n = 1 \qquad 3a_3 - a_2 - a_1 = 0 \qquad \Rightarrow \qquad a_3 = \frac{a_1}{2} + \frac{a_0}{6},$$
$$n = 2 \qquad 4a_4 - a_3 - a_2 = 0 \qquad \Rightarrow \qquad a_4 = \frac{a_1}{4} + \frac{a_0}{6}.$$

Therefore, the first terms in the power series expression for the solution y of the differential equation are given by

$$y = a_0 + a_1(x-1) + \Big(\frac{a_0}{2} + \frac{a_1}{2}\Big)(x-1)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{2}\Big)(x-1)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{4}\Big)(x-1)^4 + \cdots$$

which can be rewritten as

$$y = a_0\Big[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\Big] + a_1\Big[(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots\Big]$$

So the first four terms on each fundamental solution are given by

$$y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4,$$
$$y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4.$$

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Example 3.1.5. Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\,y = 0.$$

Solution: We then look for solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n.$$

It is convenient to rewrite the function $x\,y = [(x-2) + 2]\,y$, that is,

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} a_n x (x-2)^n \\ &= \sum_{n=0}^{\infty} a_n \big[(x-2) + 2\big](x-2)^n \\ &= \sum_{n=0}^{\infty} a_n (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n. \end{aligned} \tag{3.1.8}$$

We now relabel the first sum on the right-hand side of Eq. (3.1.8) in the following way,

$$\sum_{n=0}^{\infty} a_n (x-2)^{n+1} = \sum_{n=1}^{\infty} a_{(n-1)} (x-2)^n. \tag{3.1.9}$$

We do the same type of relabeling on the expression for y'',

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}(x-2)^n \end{aligned}$$

Then, the differential equation above can be written as follows

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{(n-1)}(x-2)^n = 0$$

$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty} \Big[(n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)}\Big](x-2)^n = 0.$$

So the *recurrence relation* for the coefficients a_n is given by

$$a_2 - a_0 = 0, \qquad (n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)} = 0, \qquad n \geqslant 1.$$

We can solve this recurrence relation for the first four coefficients,

$$\begin{aligned} n &= 0 & a_2 - a_0 &= 0 & &\Rightarrow & a_2 &= a_0, \\ n &= 1 & (3)(2)a_3 - 2a_1 - a_0 &= 0 & &\Rightarrow & a_3 &= \frac{a_0}{6} + \frac{a_1}{3}, \\ n &= 2 & (4)(3)a_4 - 2a_2 - a_1 &= 0 & &\Rightarrow & a_4 &= \frac{a_0}{6} + \frac{a_1}{12}. \end{aligned}$$

Therefore, the first terms in the power series expression for the solution y of the differential equation are given by

$$y = a_0 + a_1(x-2) + a_0(x-2)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{3}\Big)(x-2)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{12}\Big)(x-2)^4 + \cdots$$

which can be rewritten as

$$\begin{aligned} y = {} & a_0\Big[1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \cdots\Big] \\ & + a_1\Big[(x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \cdots\Big] \end{aligned}$$

So the first three terms on each fundamental solution are given by

$$y_1 = 1 + (x-2)^2 + \frac{1}{6}(x-2)^3,$$

$$y_2 = (x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4.$$

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3.1.3. The Legendre Equation. The Legendre equation appears when one solves the Laplace equation in spherical coordinates. The Laplace equation describes several phenomena, such as the static electric potential near a charged body, or the gravitational potential of a planet or star. When the Laplace equation describes a situation having spherical symmetry it makes sense to use spherical coordinates to solve the equation. It is in that case that the Legendre equation appears for a variable related to the polar angle in the spherical coordinate system. See Jackson's classic book on electrodynamics [8], § 3.1, for a derivation of the Legendre equation from the Laplace equation.