2.30

3. Second derivatives

An approach which has been found to work well for second derivatives involves applying the notion of a central difference three times. We begin with

$$f''(a) \approx \frac{f'(a+\frac{1}{2}h) - f'(a-\frac{1}{2}h)}{h}.$$

Next we approximate the two derivatives in the numerator of this expression using central differences as follows:

$$f'(a+\tfrac{1}{2}h) \approx \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad f'(a-\tfrac{1}{2}h) \approx \frac{f(a)-f(a-h)}{h}.$$

Combining these three results gives

$$f''(a) \approx \frac{f'(a+\frac{1}{2}h) - f'(a-\frac{1}{2}h)}{h}$$

$$\approx \frac{1}{h}\left\{\left(\frac{f(a+h)-f(a)}{h}\right) - \left(\frac{f(a)-f(a-h)}{h}\right)\right\}$$

$$= \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Key Point 12

Second Derivative Approximation

A central difference approximation to the second derivative $f''(a)$ is

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

2.31

Example 21

The distance x of a runner from a fixed point is measured (in metres) at intervals of half a second. The data obtained are

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use a central difference to approximate the runner's acceleration at $t = 1.5$ s.

Solution

Our aim here is to approximate $x''(t)$.
Using data with $t = 1.5$ s at its centre we obtain

$$x''(1.5) \approx \frac{x(2.0) - 2x(1.5) + x(1.0)}{0.5^2}$$

$$= -3.40 \text{ m s}^{-2}.$$

from which we see that the runner is slowing down.

Numerical Differentiation

2-32

Example// Use the forward, backward, and central difference formula to approximate the first derivative of the function $f(x) = 2e^{1.5x}$ at $x = 3$, take $h = 0.1$

Solution// $a = 3$

Forward difference $f'(x) = \frac{f(a+h) - f(a)}{h}$

$$f'(3) = \frac{f(3+0.1) - f(3)}{0.1} = \frac{f(3.1) - f(3)}{0.1} = \frac{2e^{1.5(3.1)} - 2e^{1.5(3)}}{0.1} = 291.357$$

Backward difference $f'(x) = \frac{f(a) - f(a-h)}{h}$

$$f'(3) = \frac{f(3) - f(3-0.1)}{0.1} = \frac{f(3) - f(3.1)}{0.1} = \frac{2e^{1.5(3)} - 2e^{1.5(3.1)}}{0.1} = 250.773$$

Central difference $f'(a) = \frac{f(a+h) - f(a-h)}{2h}$

$$f'(3) = \frac{f(3+0.1) - f(3-0.1)}{2*0.1} = \frac{f(3.1) - f(2.9)}{0.2}$$

$$= \frac{2e^{1.5(3.1)} - 2e^{1.5(2.9)}}{0.2} = 271.065$$

2- For the function $f(x) = x\ln x$, approximate $f''(x)$ at $x = 1$, using the second order central difference formula with $h = 0.1$.

Solution// $$f''(x) \simeq \frac{f(x_1 - h) - 2f(x_1) + f(x_1 + h)}{h^2}$$

$$\simeq \frac{f(1-0.1) - 2f(1) + f(1+0.1)}{(0.1)^2}$$

$$\simeq \frac{f(0.2) - 2f(1) + f(1.1)}{0.01} = 0.9(\ln 0.1) - 2(1\ln 1)$$

$$\simeq \frac{0.9(\ln 0.9) - 2(1\ln 1) + 1.1(\ln 1.1)}{0.01} \simeq 1.001673369$$

$$* \; f''(x) = \frac{1}{x} = \frac{1}{1} = 1$$

2.33

Exercises

1. Let $f(x) = \cosh(x)$ and $a = 2$. Let $h = 0.01$ and approximate $f'(a)$ using forward, backward and central differences. Work to 8 decimal places and compare your answers with the exact result, which is $\sinh(2)$.

2. The distance x, measured in metres, of a downhill skier from a fixed point is measured at intervals of 0.25 s. The data gathered are

t	0	0.25	0.5	0.75	1	1.25	1.5
x	0	4.3	10.2	17.2	26.2	33.1	39.1

Use a central difference to approximate the skier's velocity and acceleration at the times $t =$0.25 s, 0.75 s and 1.25 s. Give your answers to 1 decimal place.

Answers

1. Forward: $f'(a) \approx \dfrac{\cosh(a+h) - \cosh(a)}{h} = \dfrac{3.79865301 - 3.76219569}{0.01} = 3.64573199$

   Backward: $f'(a) \approx \dfrac{\cosh(a) - \cosh(a-h)}{h} = \dfrac{3.76219569 - 3.72611459}{0.01} = 3.60810972$

   Central: $f'(a) \approx \dfrac{\cosh(a+h) - \cosh(a-h)}{2h} = \dfrac{3.79865301 - 3.72611459}{0.02} = 3.62692086$

   The accurate result is $\sinh(2) = 3.62686041$.

2. Velocities at the given times approximated by a central difference are:

   20.4 m s^{-1}, 32.0 m s^{-1} and 25.8 m s^{-1}.

   Accelerations at these times approximated by a central difference are:

   25.6 m s^{-2}, 32.0 m $s^{-2}$and -14.4 m s^{-2}.