Foundation of Mathematics 1

CHAPTER 2 SETS

Dr. Bassam AL-Asadi and Dr. Emad Al-Zangana

Mustansiriyah University-College of Science-Department of Mathematics
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Chapter Two
Sets

2.1. Definitions

Definition 2.1.1. A **set** is a collection of (objects) things. The things in the collection are called **elements (member)** of the set.

A set with no elements is called **empty set** and denoted by $\emptyset$; that is, $\emptyset = \{\}$.
A set that has only one element, such as $\{x\}$, is sometimes called a **singleton set**.

List of the symbols we will be using to define other terminologies:

$|$ **or :** : such that
$\in$: an element of (belong to)
$\notin$: not an element of (not belong to)
$\subset$ **or** $\subsetneq$: a proper subset of
$\subseteq$: a subset of
$\nsubseteq$: not a subset of
$\mathbb{N}$: Set of all natural numbers
$\mathbb{Z}$: Set of all integer numbers
$\mathbb{Z}^+$: Set of all positive integer numbers
$\mathbb{Z}^-$: Set of all negative integer numbers
$\mathbb{Z}_o$: Set of all odd numbers
$\mathbb{Z}_e$: Set of all even numbers
$\mathbb{Q}$: Set of all rational numbers
$\mathbb{R}$: Set of all real numbers

Set Descriptions 2.1.2.

(i) Tabulation Method
The elements of the set listed between commas, enclosed by braces.
(1) $\{1,2,37,88,0\}$
(2) $\{a, e, i, o, u\}$ Consists of the lowercase vowels in the English alphabet.
(3) $\{\dots, -4, -2, 0, 2, 4, 6\}$ Continue from left side
$\{-4, -2, 0, 2, 4, 6, \dots\}$ Continue from right side
$\{\dots, -4, -2, 0, 2, 4, 6, \dots\}$ Continue from left and right sides.
(4) $B = \{\{2,4,6\}, \{1,3,7\}\}$.

(ii) Rule Method
Describe the elements of the set by listing their properties writing as

$$S = \{x |\, A(x)\},$$

where $A(x)$ is a statement related to the elements x. Therefore,

$$x \in S \Leftrightarrow A(x) \text{ is hold}$$

(1) $A = \{x | x \text{ is a positive integers and } x > 10\}$
$A = \{x | x \in \mathbb{Z}^+ \text{and } x > 10\}$.
(2) $\mathbb{Z}_o = \{x | x = 2n - 1 \text{ and } n \in \mathbb{Z}\}$
$= \{2n - 1 |\, n \in \mathbb{Z}\}$.
(3) $\{x \in \mathbb{Z} |\, |x| < 4\} = \{-3, -2, -1, 0, 1, 2, 3\}$.
(4) $\{x \in \mathbb{Z} |\, x^2 - 2 = 0\} = \emptyset$.

Examples 2.1.3.
(i) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ Integer numbers.
(ii) $\mathbb{Z}_e = \{x | x = 2n \text{ and } n \in \mathbb{Z}\}$
$= \{2n |\, n \in \mathbb{Z}\}$. Even numbers
Note that 2 is an element of $\mathbb{Z}_e$ so, we write $2 \in \mathbb{Z}$. But, $5 \notin \mathbb{Z}_e$.

(iii) Let C be the set of all natural numbers which are less than 0.
In this set, we observe that there are no elements. Hence, C is an empty set; that is,

$$C = \emptyset.$$

Definition 2.1.4.
(i) A set A is said to be a **subset** of a set B if every element of A is an element of B and denote that by $A \subseteq B$. Therefore,

$$A \subseteq B \Leftrightarrow \forall x (x \in A \Longrightarrow x \in B).$$

(ii) If A is a nonempty subset of set B and B contains an element which is not a member of A, then A is said to be **proper subset** of B and denoted this by $A \subset B$ or $A \subsetneq B$; that is, A is said to be a **proper subset** of B if and only if
(1) $A \neq \emptyset$, (2) $A \subset B$ and (2) $A \neq B$.

We use the expression $A \nsubseteq B$ means that A is **not** a subset of B.

Examples 2.1.5.
(i) An empty set $\emptyset$ is a subset of any set B; that is, for every set B, $\phi \subseteq B$.

If this were not so, there would be some element $x \in \emptyset$ such that $x \notin B$. However, this would contradict with the definition of an empty set as a set with no elements.

(ii) Let B be the set of natural numbers. Let A be the set of even natural numbers. Clearly, A is a subset of B. However, B is not a subset of A, for $3 \in B$, but $3 \notin A$.

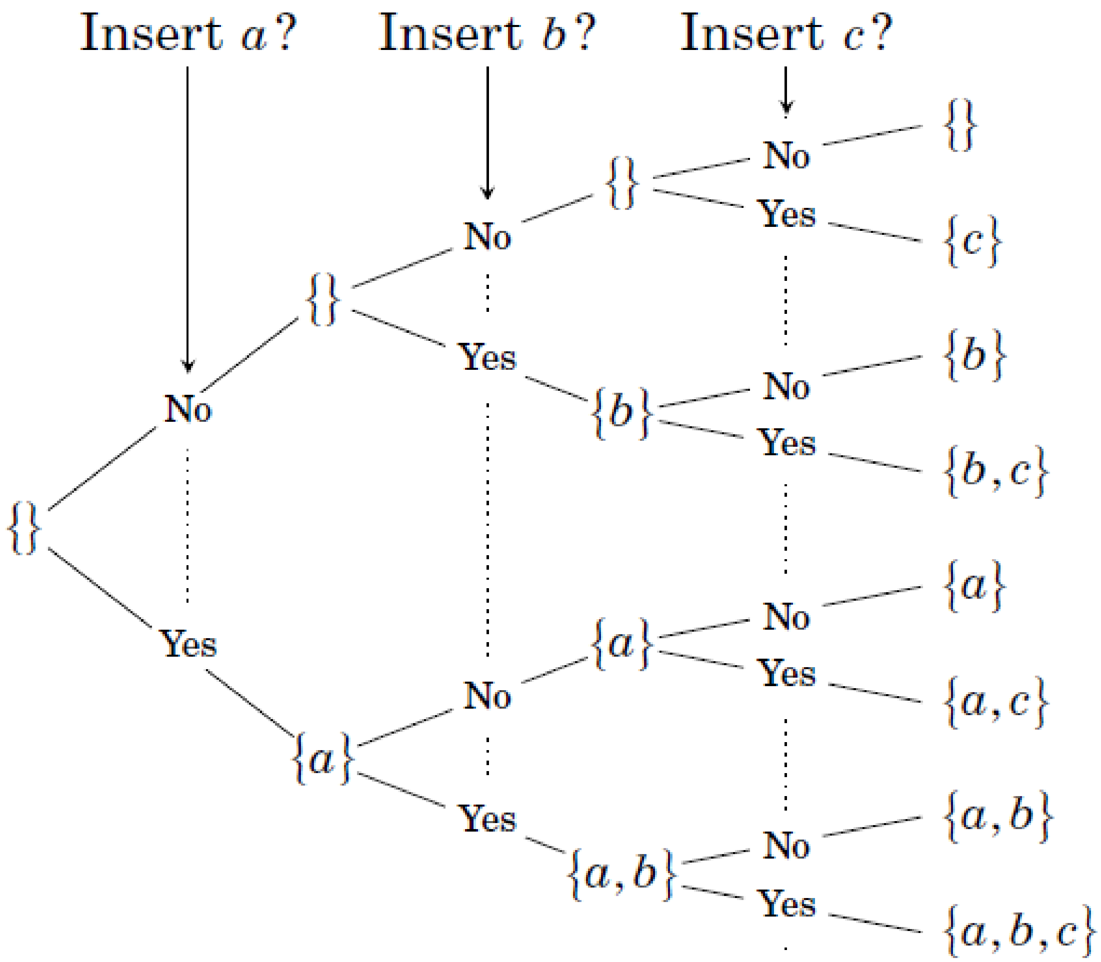
Theorem 2.1.6. (Properties of Sets)
Let A, B, and C be sets.
(i) For any set $A, A \subseteq A$. (Reflexive Property)
(ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (Transitive Property)

Proof.
(ii)

1	$(A \subseteq B) \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$	Hypothesis and Def. $\subseteq$
2	$(B \subseteq C) \Leftrightarrow \forall x(x \in B \Rightarrow x \in C)$	Hypothesis and Def. $\subseteq$
	$\Rightarrow \forall x(x \in A \Rightarrow x \in C)$	Inf. (1),(2) Syllogism Law
	$\Leftrightarrow A \subseteq C$	Def. of $\subseteq$

Definition 2.1.7 If X is a set, the **power set** of X is another set, denoted as $P(X)$ and defined to be the set of all subsets of X. In symbols,
$$P(X) = \{A | A \subseteq X\}.$$
That is, $A \subseteq X$ if and only if $A \in P(X)$.

Example 2.1.8.
(i) $\emptyset$ and a set X are always members of $P(X)$.
(ii) suppose $X = \{a, b, c\}$. Then
$$P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$
The way to finding all subsets of X is illustrated in the following figure.



From the above example, if a finite set X has n elements, then it has $\mathbf{2^n}$ subsets, and thus its power set has $\mathbf{2^n}$ elements.

(iii) $P(\{1,2,4\}) = \{\emptyset, \{0\}, \{1\}, \{4\}, \{0,1\}, \{0,4\}, \{1,4\}, \{1,2,4\}\}$.
(iv) $P(\emptyset) = \{\emptyset\}$.
(v) $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
(vi) $P(\{\mathbb{Z}, \mathbb{R}\}) = \{\emptyset, \{\mathbb{Z}\}, \{\mathbb{R}\}, \{\mathbb{Z}, \mathbb{R}\}\}$.

The following are wrong statements.

(v) $P(1) = \{\emptyset, \{1\}\}$.
(vi) $P(\{1, \{1,2\}\}) = \{\ \emptyset, \{1\}, \{1,2\}, \{1, \{1,2\}\}\ \}$.
(vii) $P(\{1, \{1,2\}\}) = \{\ \emptyset, \{\{1\}\}, \{\{1,2\}\}, \{1, \{1,2\}\}\ \}$.

2.2. Equality of Sets

Definition 2.2.1. Two sets, A and B, are said to be **equal** if and only if A and B contain exactly the same elements and denote that by $A = B$. That is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
The description $A \neq B$ means that A and B are not equal sets.

Example 2.2.2.
Let $\mathbb{Z}_e$ be the set of even integer numbers and $B = \{x | x \in \mathbb{Z}$ and divisible by 2$\}$.
Then $\mathbb{Z}_e = B$.
Proof.
To prove $\mathbb{Z}_e \subseteq B$.
$\mathbb{Z}_e = \{2n | n \in \mathbb{Z}\}$.

$x \in \mathbb{Z}_e \Leftrightarrow \exists\, n \in \mathbb{Z} : x = 2n$ Def. of $\mathbb{Z}_e$.

$\Longrightarrow \frac{x}{2} = n$ Divide both side of $x = 2n$ by 2.

$\Longrightarrow x \in B$ Def. of B.

(1) $\Longrightarrow \mathbb{Z}_e \subseteq B$ Def. of subset.

To prove $B \subseteq \mathbb{Z}_e$.

$x \in B \Leftrightarrow \exists\, n \in \mathbb{Z} : \frac{x}{2} = n$ Def. of $\mathbb{Z}_e$.

$\Longrightarrow x = 2n$ Multiply $\frac{x}{2} = n$ by 2.

$\Longrightarrow x \in \mathbb{Z}_e$ Def. of $\mathbb{Z}_e$.

(2) $\Longrightarrow B \subseteq \mathbb{Z}_e$ Def. of subset.

$\mathbb{Z}_e = B$ inf (1),(2) and def. of equality.

Remark 2.2.3.
(i) Two equal sets always contain the same elements. However, the rules for the sets may be written differently, as in Example 2.2.2.
(ii) Since any two empty sets are equal, therefore, there is a unique empty set.
(iii) the symbols $\subseteq, \subset, \subsetneq, \nsubseteq$ are used to show a relation between two sets and not between an element and a set. With one exception, if x is a member of a set A, we may write $x \in A$ or $\{x\} \subseteq A$, but **not** $x \subseteq A$.
(iv) $\phi \neq \{\phi\}$.

Theorem 2.2.4. (Properties of Set Equality)
(i) For any set $A, A = A$. (Reflexive Property)
(ii) If $A = B$, then $B = A$. (Symmetric Property)
(iii) If $A = B$ and $B = C$, then $A = C$. (Transitive Property)

Definition 2.2.5. Let A and B be subsets of a set X. The **intersection** of A and B is the set

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\},$$

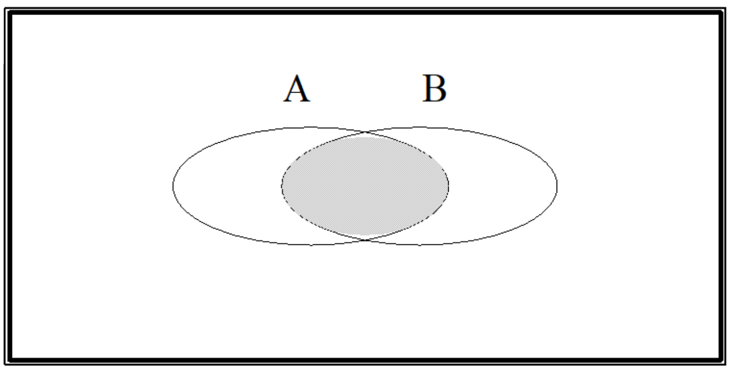
or

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}.$$

Therefore, $A \cap B$ is the set of all elements in common to both A and B.

Example 2.2.6.
(i) Given that the box below represents X, the shaded area represents $A \cap B$:

X:



(ii) Let $A = \{2,4,5\}$ and $B = \{1,4,6,8\}$. Then, $A \cap B = \{4\}$.

(iii) Let $A = \{2,4,5\}$ and $B = \{1,3\}$. Then $A \cap B = \emptyset$.

Definition 2.2.7. If two sets, A and B are two sets such that $A \cap B = \emptyset$ we say that A and B are **disjoint.**

Definition 2.2.8. Let A and B be two subsets of a set X. The **union** of A and B is the set

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\},$$

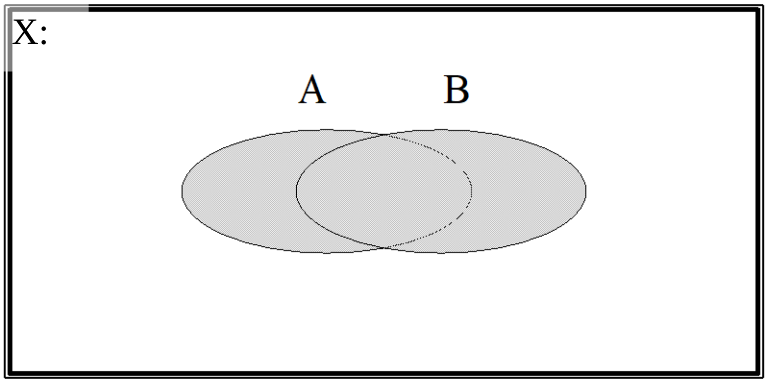
or

$$A \cup B = \{x \in X \mid x \in A \vee x \in B\}.$$

Therefore, $A \cup B$ = the set of all elements belonging to A or B.

Example 2.2.9.

(i) Given that the box below represents X, the shaded area represents $A \cup B$:



(ii) Let $A = \{2,4,5\}$ and $B = \{1,4,6,8\}$. Then, $A \cup B = \{1,2,4,5,6,8\}$.
(iii) $\mathbb{Z}_e \cup \mathbb{Z}_o = \mathbb{Z}$.

Remark 2.2.10.
It is easy to extend the concepts of intersection and union of two sets to the intersection and union of a finite number of sets. For instance, if $X_1, X_2, \dots, X_n$ are sets, then

$$X_1 \cap X_2 \cap \dots \cap X_n = \{x | x \in X_i \text{ for all } i = 1, \dots, n\}$$

and

$$X_1 \cup X_2 \cup \dots \cup X_n = \{x \mid x \in X_i \text{ for some } i = 1,2, \dots, n\}.$$

Similarly, if we have a collection of sets $\{X_i : i = 1,2, \dots\}$ indexed by the set of positive integers, we can form their intersection and union. In this case, the intersection of the X_i is

$$\bigcap_{i=1}^{\infty} X_i = \{x \in X_i \text{ for all } i = 1,2, \dots\}$$

and the union of the X_i is

$$\bigcup_{i=1}^{\infty} X_i = \{x \in X_i \text{ for some } i = 1,2, \dots\}.$$

Theorem 2.2.11. Let A, B, and C be arbitrary subsets of a set X. Then

(i) $A \cap B = B \cap A$ (Commutative Law for Intersection)

$A \cup B = B \cup A$ (Commutative Law for Union)

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative Law for Intersection)
$A \cup (B \cup C) = (A \cup B) \cup C$ (Associative Law for Union)
(iii) $A \cap B \subseteq A$
(iv) $A \cap X = A$; $A \cup \emptyset = A$
(v) $A \subseteq A \cup B$
(vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law of Union with respect to Intersection).
(vii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law of Intersection with respect to Union),
(viii) $A \cup A = A, A \cap A = A$ (Idempotent Laws)
(ix) $A \cup \emptyset = A, A \cap X = A$ (Identity Laws)
(x) $A \cup X = X,\ A \cap \emptyset = \emptyset$ (Domination Laws)
(xi) $A \cup (A \cap B) = A$ (Absorption Laws) $A \cap (A \cup B) = A$.

Proof.
(i) $A \cap B = B \cap A$. This proof can be done in two ways.

The *first proof*
Uses the fact that the two sets will be equal only if
$(A \cap B) \subseteq (B \cap A)$ and $(B \cap A) \subseteq (A \cap B)$.

(1) Let x be an element of $A \cap B$

Therefore, $x \in A \wedge x \in B$	Def. of $\cap$
Thus, $x \in B \wedge x \in A$	Commutative Property of $\wedge$
Hence, $x \in B \cap A$	Def. of $B \cap A$
Therefore, $A \cap B \subseteq B \cap A$	Def. of $\subseteq$

(2) Let x be an element of $B \cap A$

Therefore, $x \in B \wedge x \in A$	Def. of $\cap$
Thus, $x \in A \wedge x \in B$	Commutative property of $\wedge$
Hence, $x \in A \cap B$	Def. of $\cap$
Thus, $B \cap A \subseteq A \cap B$	Def. of $\subseteq$

Therefore, $A \cap B = B \cap A$ Inf. (1),(2)

The *second proof*

$$\begin{aligned} A \cap B &= \{x \mid x \in A \cap B\} && \\ &= \{x \mid x \in A \wedge x \in B\} && \text{Def. of } \cap \\ &= \{x \mid x \in B \wedge x \in A\} && \text{Commutative property of } \wedge \\ &= \{x \mid x \in B \cap A\} && \text{Def, of } \cap \\ &= B \cap A && \end{aligned}$$

(iii) $(A \cap B) \subseteq A$

It must be shown that each element of $A \cap B$ is an element of A.

Let $x \in A \cap B$
Thus, $x \in A \wedge x \in B$ Def. of $\cap$
Hence, $x \in A$

Therefore, $(A \cap B) \subseteq A$

(iv) $A \cap X = A$

(1) $A \cap X \subset A$ Inf. (iii) above

(2) Let $x \in A$
Thus, $x \in X$ $A \subseteq X$ is given
Hence, $x \in A \wedge x \in X$ Def. of $\wedge$
Therefore, $x \in A \cap X$ Def. of $\cap$
Thus, $A \subseteq A \cap X$ Def. of $\subseteq$

Thus, $A \cap X = A$ Inf. (1),(2)

Definition 2.2.12. Let A and B be subsets of a set X. The set $B - A$, called the **difference** of B and A, is the set of all elements in B which are not in A.
Thus,

$$B - A = \{x \in X \mid x \in B \text{ and } x \notin A\}.$$

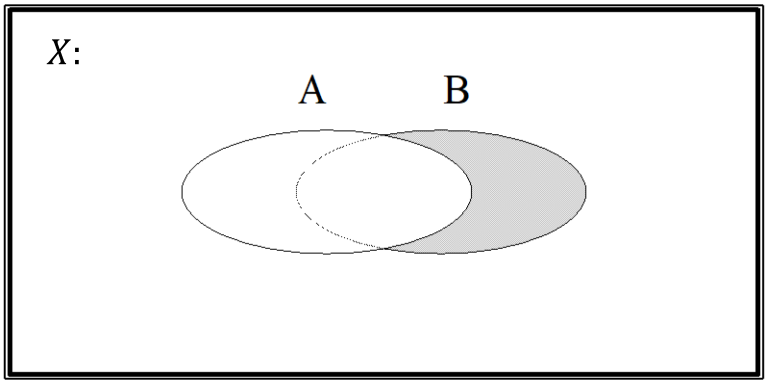
Example 2.2.13.

(i) Let $B = \{2,3,6,10,13,15\}$ and $A = \{2,10,15,21,22\}$.Then

$$B - A = \{3,6,13\}.$$

(ii) $\mathbb{Z} - \mathbb{Z}_o = \mathbb{Z}_e$.

(iii) Given that the box below represents X, the shaded area represents $B - A$.

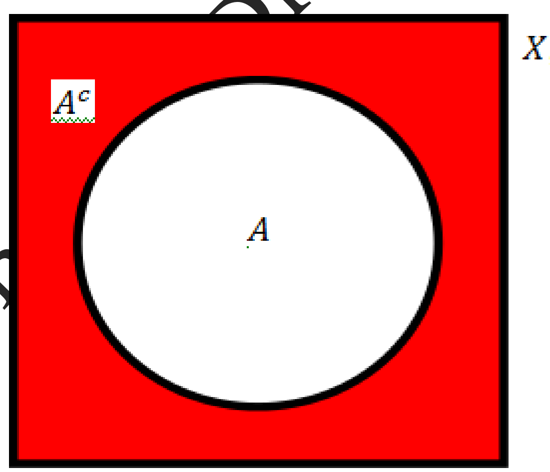


Theorem 2.2.14. Let A and B be subsets of a set X. Then
(i) $A - A = \emptyset,\ A - \emptyset = A$ and $\emptyset - A = \emptyset$

Definition 2.2.15. If $A \subseteq X$, then $X - A$ is called the **complement** of A with respect to X and denoted that by the symbol

$$X \backslash A \text{ or } A^c.$$

Thus, $A^c = \{x \in X \mid x \notin A\}$.



Theorem 2.2.16. Let A and B be subsets of a set X. Then
(i) $A^{cc} = A$.
(ii) $X^c = \emptyset$; $\emptyset^c = X$.
(iii) $A \cup A^c = X,\ A \cap A^c = \emptyset$ (Inverse Laws)
(iv) If $A \subseteq B$, then $B^c \subseteq A^c$.
(v) $A \cap B = \emptyset \Leftrightarrow A \subseteq B^c$.
Proof. Exercise.

Theorem 2.2.17. Let A and B be subsets of a set X. Then

(i) $\left.\begin{array}{l}(A\cup B)^c = A^c\cap B^c\\ (A\cap B)^c = A^c\cup B^c\end{array}\right\},$ (De Morgan's Law)

(ii) Let A and B be subsets of a set X. Then, $A - B = A\cap B^c$.

(iii) $A^c - B^c = B - A$.

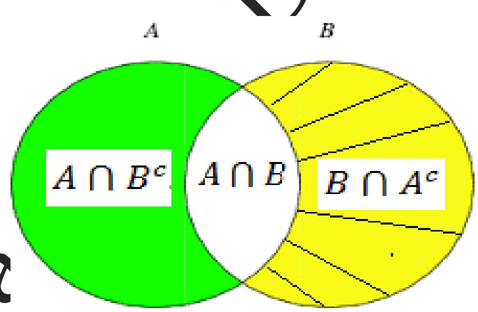
Proof.

(i) Let $x \in (A\cup B)^c$

$\Leftrightarrow x \notin (A\cup B)$ Def. of complement
$\Leftrightarrow \sim(x \in A\cup B)$ Def. of $\notin$
$\Leftrightarrow \sim(x \in A \vee x \in B)$ Def. of A ∪ B
$\Leftrightarrow \sim(x \in A)\wedge \sim(x\in B)$ De Morgan's Law
$\Leftrightarrow x\notin A \wedge x \notin B$ Def of $\notin$
$\Leftrightarrow x \in A^c \wedge x \in B^c$ Def. of complement
$\Leftrightarrow x \in A^c\cap B^c$ Def. of $\cap$

Hence $(A\cup B)^c = A^c\cap B^c$.

(ii) $A - B = \{x \in X \mid x \in A \text{ and } x \notin B\}$
$= \{x \in X \mid x \in A \text{ and } x \in B^c\}$ Def. of complement of B^c
$= A\cap B^c$ Def. of complement intersection



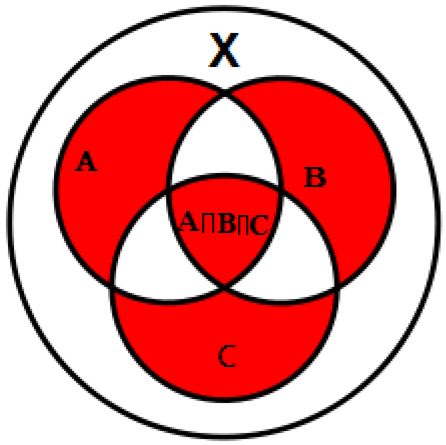
(iii) Exercise.

Definition 2.2.18. Let A and B be subsets of a set X. The set

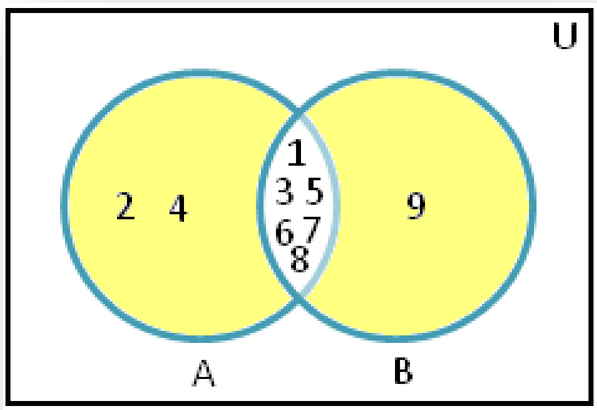
$$A \triangle B = (A-B)\cup(B-A)$$

is called the **symmetric difference.**

Sometimes the symbol $A\oplus B$ is used for symmetric difference.



Example 2.2.19. Let $A = \{1,2,3,4,5,6,7,8\}$ and $B = \{1,3,5,6,7,8,9\}$ are subsets of $U = \{1,2,3,4,5,6,7,8,9,10\}$.
$A - B = \{2,4\}$
$B - A = \{9\}$
$A \triangle B = (A - B) \cup (B - A) = \{2,4,9\}$.



A △ B

Theorem 2.2.20. Let A, B and C are subsets of X. Then

(i) $A \triangle \emptyset = A$.
(ii) $A \triangle B = \emptyset \Leftrightarrow A = B$.
(iii) $A \triangle B = B \triangle A$.
(iv) $A \triangle A = \emptyset$.

Proof. Exercise.

Theorem 2.2.21. (Properties of $\cup, \cap, -, \triangle$ and $P(X)$)

(i) $A-(B\cap C)=(A-B)\cup(A-C)$ De Morgan's Low on $-$
$A-(B\cup C)=(A-B)\cap(A-C)$.

(ii) $A-(A\cap B)=(A-B)=(A\cup B)-B$
$A-(A\cup B)=\emptyset$.

(iii) $(A\cap B)-C=(A-C)\cap(B-C)$
$(A\cup B)-C=(A-C)\cup(B-C)$.

(iv) $(A-B)\cap(C-D)=(C-B)\cap(A-D)$.

(v) If $A\subseteq B$, then $P(A)\subseteq P(B)$.

(vi) $P(A\cap B)=P(A)\cap P(B)$.

(vii) $P(A)\cup P(B)\subseteq P(A\cup B)$. The converse is not true.

(viii) $A=B \Leftrightarrow P(A)=P(B)$.

(ix) $A\cap B=\emptyset \Leftrightarrow P(A)\cap P(B)=\emptyset$.

(x) $A\triangle B=(A\cup B)-(A\cap B)$.

(xi) $A\triangle(B\triangle C)=(A\triangle B)\triangle C$. Associative Law of $\triangle$

(xii) $A\triangle C=B\triangle C \implies A=B$.

(xiii) If $A\subseteq B$ and $C=B-A$, then $A=B-C$.

(xiv) $A\cap(B-C)=(A\cap B)-(A\cap C)$.

(xv) $(A-B)\cap(C-D)=(A\cap C)-(B\cup D)$.

(xvi) $A\cap(B\triangle C)=(A\cap B)\triangle(A\cap C)$. Dist. of $\cap$ on $\triangle$

Proof.

(i) $A-(B\cap C)=A\cap(B\cap C)^c$ Theorem 2.2.17(ii)
$=A\cap(B^c\cup C^c)$ De Morgan's Law
$=(A\cap B^c)\cup(A\cap C^c)$ Dist. Law
$=(A-B)\cup(A-C)$. Theorem 2.2.17(ii)

(vii) Let $H\in P(A)\cap P(B)$
$\implies H\in P(A)\wedge H\in P(B)$ Def. $\cap$
$\implies H\subseteq A\wedge H\subseteq B$ Def. of power set
$\implies H\subseteq(A\cap B)$ Def. $\cap$
$\implies H\in P(A\cap B)$ Def. of power set

(x) $x\in A\triangle B \Leftrightarrow x\in(A-B)\cup(B-A)$ Def. of $\triangle$
$\Leftrightarrow x\in(A-B)\vee(B-A)$ Def. $\cup$
$\Leftrightarrow x\in A\wedge x\notin B)\vee(x\in B\wedge x\notin A)$ Def. of difference
$\Leftrightarrow (x\in A\vee x\in B)\wedge(x\notin B\vee x\in B)$ Dist. Law of
$\wedge$
$(x\in A\vee\notin A)\wedge(x\notin B\vee x\notin A)$

$\Leftrightarrow$	$(x \in A \lor x \in B) \land T$ $\land$ $T \land (x \notin B \lor x \notin A)$	Tautology
$\Leftrightarrow$	$x \in A \lor x \in B$ $\land$ $x \in B^c \lor x \in A^c$	Identity Law of $\land$
$\Leftrightarrow$	$x \in (A \lor B)$ $\land$ $x \in (B^c \lor A^c)$	
$\Leftrightarrow$	$x \in (A \cup B)$ $\cap$ $x \in (B^c \cup A^c) = (A \cap B)^c$	Def. of $\cup$ and De Morgan's Law
$\Leftrightarrow$	$x \in (A \cup B) \cap (A \cap B)^c$	
$\Leftrightarrow$	$x \in (A \cup B) - (A \cap B)$	Theorem 2.2.17(ii)