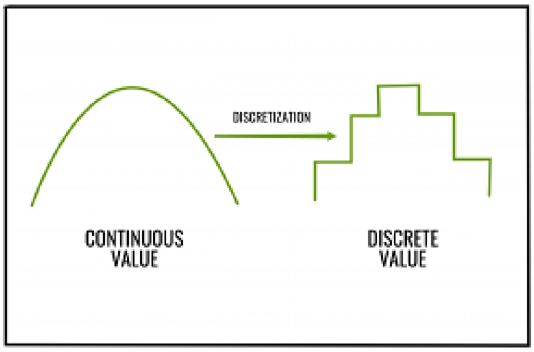
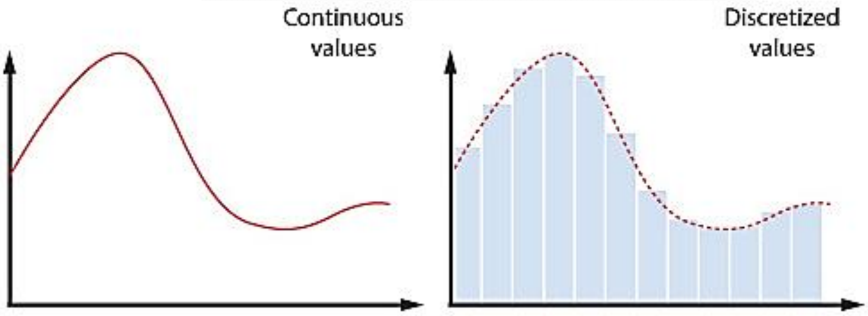
Lecture (1)

Discretization Error and Convergence

1.1 Discretization Error

Discretization is the process of transferring continuous functions, models, variables, and equations into discrete counterparts. This process is usually carried out as a first step toward making them suitable for numerical evaluation and implementation on digital computers.





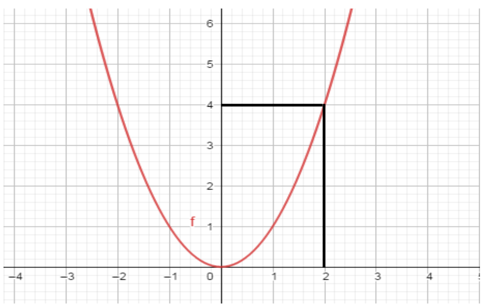
(infinite values → finite values)

Ex: Integrate the following function analytically and numerically $f(x) = x^2$

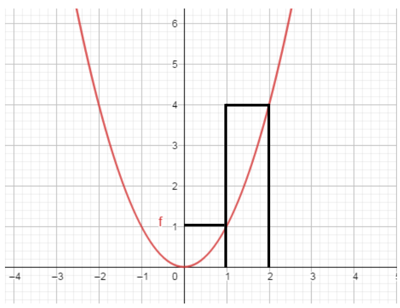
Sol. (1) Analytically: $\int f(x)dx = \int x^2\, dx = \frac{x^3}{3} + C$

$$from\ x = 0\ to\ x = 2\ the\ solution\ is = \frac{2^3}{3} = 2.666$$

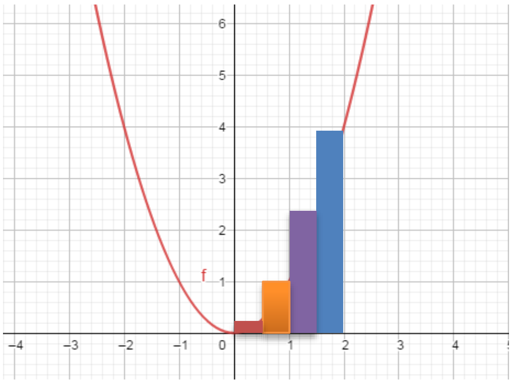
(2) Numerically (Area under the curve)



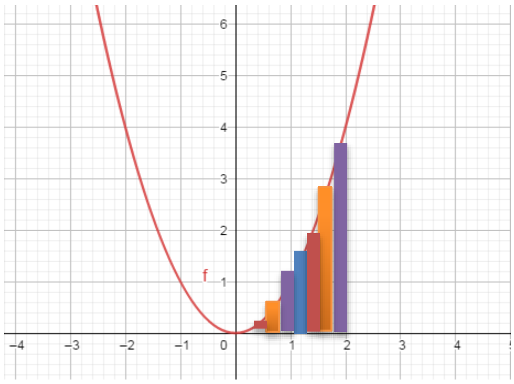
Area= 8



Area = 5



Area = 3.77



Area = 2.97

As we can see, the solution is approaching the exact value (which is the analytical solution) each time we add more fine rectangles (less Δx)

1.2 Truncation Error

Truncation errors are the errors that result from using an approximation in place of an exact mathematical procedure.

$$e^x = \underbrace{\overbrace{\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}}}^{\text{Approximation}} + \overbrace{\boxed{\frac{x^{n+1}}{(n+1)!} + \dots}}^{\text{Truncation Errors}}}_{\text{Exact mathematical formulation}}$$

Analytical solutions are logical procedures that yield an **exact solution**.

Numerical solutions come from discrete equations that give approximate solutions.

The truncation error can be made as small as we wish by making Δx and Δt smaller and smaller, as long as the system is consistent.

CONSISTENCY: A **finite difference approximation** is considered consistent if by reducing the mesh and time step size, the **truncation error** terms could be made to approach zero. In that case, the solution to the difference equation would approach the true solution to the PDE.

Question: Explain each of Differential Equation and Difference Equation.

There is another source of errors is the infinite accuracy of computers. This round off error is a feature of the computers and the program. Sometimes this error causes problems but not often, we will not worry about it for now.

We should take into account the error in solving the solution in discrete equation, that is, the difference between the solution of discrete equation and the solution of continuous differential equation، i.e.

$$A(i,j) - A(i\Delta x, j\Delta t)$$

This error is called the Discretization Error. Unfortunately, a decrease in the truncation error does not necessarily confirm a decrease in the discretization error. It is even possible that the decrease in the truncation error to be accompanied by an increase in the discretization error!

We will now analyze the changes of the discretization error with smoothing of the grid (that is, as Δx and Δt get closer to zero). If the discretization error approaches zero, then we say that the solution *converges*. Figure (1.1) illustrates a situation in which the solution does not converge with further smoothing of the mesh. The diagonal line in the figure shows the feature along which A is spread, that is, A is constant along the line. This line represents the exact solution (the analytical solution).

To work on numerical approximation for this solution, we first choose Δx and Δt so that the grid points are the bold points in the figure. The set of grid points with A values and on which $A_{i,j}$ is based is called the domain of dependence. The shaded area in the figure shows the dependency for an upstream scheme described by the equation

$$\frac{A_{i,j+1} - A_{i,j}}{\Delta t} + c\left(\frac{A_{i,j} - A_{i-1,j}}{\Delta x}\right) = 0 \qquad (1.1)$$

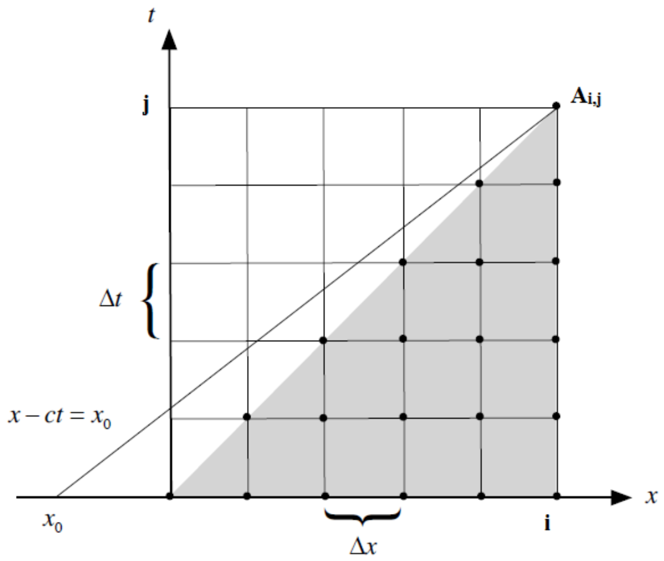


Figure (1.1) the shaded area in the figure represents a domain of dependency for an upstream scheme in the point ($\boldsymbol{i\Delta x, j\Delta t}$).

We can increase the accuracy of the scheme by divide by halve $\Delta x \; and \; \Delta t$ but the dependence domain does not change as long as the ratio $c\frac{\Delta t}{\Delta x}$ remains constant. This is evidence that $c\frac{\Delta t}{\Delta x}$ is an important quantity.
Suppose that the line that passes through the point (iΔx, jΔt) is $x - ct = x_0$, where x_0 is a constant quantity, which does not fall within the domain of dependence.

In general, it is not hoped to obtain a smaller discretization error, no matter how much the values of Δx and Δt change as long as the quantity $\frac{c\Delta t}{\Delta x}$ is unchanged, because the true solution (the true numerical solution) depends only on the initial value of A in the individual point $(x_0, 0)$ that cannot influence $A_{i,j}$.

You can change $A(x_0, 0)$ [and hence the exact solution $A\ (i\Delta x, j\Delta t)$], but the computed solution $A_{i,j}$ will remain the same. In such a case, the error in the solution will not be diminished by the smoothing of the grid. This shows that if the value of c is such that x_0 falls outside the domain of dependency, then it is not possible for the solution of the finite difference equation (numerical solution) to get close to the solution of the differential equation (analytical), whether the grid becomes smooth or not. The finite difference equation converges to the differential equation, but the solution to the differential equation does not converge to the solution of the differential equation. The truncation error goes to zero but the discretization error does not.

The above discussion shows that:

$$0 \le \frac{c\Delta t}{\Delta x} \le 1 \qquad (1.2)$$

is a necessary condition for convergence in upstream scheme

Note that if c is negative (giving the downstream scheme). The distinguished line is outside the domain of dependence in the figure. Actually, if $c < 0$, we can get:

$$\frac{A_{i,j+1} - A_{i,j}}{\Delta t} + c\left(\frac{A_{i-1,j} - A_{i,j}}{\Delta x}\right) = 0 \qquad (1.3)$$

instead of the upstream scheme equation (1.1).

In summary: the truncation error measures the approximation accuracy of the differential operator. It is a measure for the accuracy the terms of the differential equation are approximated. Reducing the truncation error to acceptable levels is usually easy, but reducing the discretization error is more difficult.