Chapter 6

Laplas Transformation:

هذا الفصل مهم جداً

domain الدالة الجديدة F(s) تمثل كل القيم

الجديدة ويكون معرف فقط بالقيم التي تجعل $\int$

exists.

نحل التكامل ثم ناخذ limit

Laplace Transform:

Definitions:

Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of $f(t)$ is the function $F(s)$ defined by the integral $\mathcal{L}\{f\}_{(s)} = F(s) := \int_0^\infty e^{-st} \cdot f(t)\, dt$ ---- ① → تعريف مهم

The domain of $F(s)$ is all the values of s for which the integral in ① excists. The Laplace transform of $f(t)$ is denoted by both $F(s)$ and $\mathcal{L}\{f\}_{(s)}$.

Notice That the integral in ① is an improper integral that is defined by $\int_0^\infty e^{-st} f(t)\,dt := \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$ where the limit exists.

Example ①

: Determine the Laplace transform of the constants function $f(t) = 1$, $t \geq 0$.

solution:

: Using the definition of the transform, we compute:

$$\mathcal{L}\{f\}_{(s)} = F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-st} \cdot 1\, dt = \lim_{N\to\infty} \int_0^N e^{-st}\, dt = \lim_{N\to\infty} \left. \frac{e^{-st}}{-s} \right]_0^N$$

$e^{-\infty} \to 0$ (+) بشرط

$$= \lim_{N\to\infty} \left[\cancel{\frac{-e^{-sN}}{s}} + \frac{1}{s} \right] = \frac{1}{s}$$

$$F(s) = \frac{1}{s} \quad s > 0$$

لازم موجبة

When $s \le 0$ The integral $\int_0^\infty e^{-st}\,dt$ div. why?

Hence $F(s) = \frac{1}{s}$ with The domain of $F(s)$ being all $s > 0$.

Example 2,

Determine The Laplace transform of The constant function $f(t) = e^{at}$, $t \ge 0$

Solution:

using The defintion of The transform, we compute:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

$$= \lim_{N \to \infty} \int_0^N e^{-st} e^{at}\,dt = \lim_{N \to \infty} \int_0^N e^{-(s-a)t}\,dt = \lim_{N \to \infty} \left. \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^N = \frac{1}{s-a}$$

Note:

Ex ① constant function $= 1 = e^{0t}$ } transform $= \frac{1}{s}$

Ex ② $f(t) = e^{at}$ } $= \frac{1}{s-a}$

e^{3t} H.W تحويلها $\frac{1}{s-3}$

Example 3:-

Find $\mathcal{L}\{\sin bt\}$, where b is a non Zero constant

Solution:

using The defintion of The transform, we compute

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt.$$

$$= \lim_{N \to \infty} \int_0^N e^{-st} \cdot \sin bt\,dt. \qquad u \cdot dv$$

using integration by parts twice, we find:

$$\mathcal{L}\{\sin bt\}(s) = \lim_{N \to \infty} \left[\left. \frac{-e^{-st}}{s^2+b^2} (s \sin bt + b\cos bt) \right]_0^N \right]$$

$$= <$$

$$= \lim_{N \to \infty} \left[\frac{b}{s^2+b^2} - \frac{e^{-sN}}{s^2+b^2} (s \sin bN + b \cos bN) \right]$$

$$= \frac{b}{s^2+b^2} \quad \text{for } s > 0.$$

H.W.:

Determine The Laplace transform of $f(t) = \begin{cases} 2 & 0 < t < 5 \\ 0 & 5 < t < 10 \\ e^{4t} & 10 < t \end{cases}$

يوجد تصحيح للحل.

$$\mathcal{L}\{2\} + \mathcal{L}\{0\} + \mathcal{L}\{e^{4t}\}$$

$$= \int_0^5 e^{-st} \cdot 2\, dt + \int_5^{10} e^{-st} \cdot 0\, dt + \int_{10}^{\infty} e^{-st} \cdot e^{4t}\, dt.$$

$$= \frac{2}{-s} e^{-st} \Big|_0^5 + 0 + \lim_{N \to \infty} \int_{10}^{N} e^{(-s+4)t}\, dt$$

$$= \frac{2}{s} e^{-st} + \frac{2}{s} + \lim_{N \to \infty} \frac{1}{-s+4} e^{(-s+4)t} \Big|_{10}^{N}$$

$$= \frac{2}{s} \left[-e^{-5s} + 1 \right] + \frac{+1}{s-4} e^{(-s+4)10}$$

Some important properties of Laplace transform:

Linearity:

An important property of The Laplace transform is its linearity. That is, The Laplace transform $\mathcal{L}$ is a linear operator.

Theorem:

Let f_1 and f_2 be functions whose Laplace transform exists for $s>0$, and let c be a constant. Then:

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linearity

$$\begin{cases} \mathcal{L}\{f_1+f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\} \\ \mathcal{L}\{cf_1\} = c\,\mathcal{L}\{f_1\} \end{cases}$$

Proof:

Using The linearity properties of integration we have for $s>0$

$$\mathcal{L}\{f_1+f_2\}(s) = \int_0^\infty e^{-st}\left[f_1(t)+f_2(t)\right]dt$$

$$= \int_0^\infty e^{-st} f_1(t)\,dt + \int_0^\infty e^{-st} f_2(t)\,dt.$$

$$= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}.$$

In a similar fashion $\mathcal{L}\{cf_1\}(s) = \int_0^\infty e^{-st}\left[cf_1(t)\right]dt$

$$= c\int_0^\infty e^{-st} f_1(t)\,dt = c\,\mathcal{L}\{f_1(t)\}$$

Example:

Determin $\mathcal{L}\{11+5e^{4t}-6\sin 2t\}$

Solution:

$$\mathcal{L}\{11+5e^{4t}-6\sin 2t\} = \mathcal{L}\{11\} + \mathcal{L}\{5e^{4t}\} - \mathcal{L}\{6\sin 2$$

$= 11\,\mathcal{L}\{1\} + 5\,\mathcal{L}\{e^{4t}\} - 6\,\mathcal{L}\{\sin 2t\}.$

$= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}$

$\mathcal{L}\{1\}(s) = \frac{1}{s}$

$\mathcal{L}\{e^{4t}\}(s) = \frac{1}{s-4}$

$\mathcal{L}\{\sin 2t\}(s) = \frac{2}{s^2+2^2}$

from The previous Examples 1, 2, 3.

H.W:-

Determin $\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}.$

$= \mathcal{L}\{4t^2\} - \mathcal{L}\{3\cos 2t\} + \mathcal{L}\{5e^{-t}\}$

$= 4\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\}$

$= \frac{8}{t^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$

$F(s) = \lim_{N\to\infty} \int_0^N e^{-st} t^2\, dt = \frac{t^2 e^{-st}}{-s} + 2\int t \frac{e^{-st}}{s}\, dt$

$u = t^2 \Rightarrow du = 2t\,dt$

$dv = e^{-st}dt \longrightarrow v = \frac{e^{-st}}{-s}$

$= \frac{t^2 e^{-st}}{-s} - 2\frac{t e^{-st}}{s^2} + \frac{2e^{-st}}{s^3}$

$u = t \rightarrow du = dt$

$dv = \frac{e^{-st}}{s}dt \longrightarrow v = \frac{-e^{-st}}{s^2}$

$= \lim_{N\to\infty} \underbrace{\frac{N^2 e^{-sN}}{-s}}_{0} - 2\underbrace{\frac{N e^{-sN}}{s^2}}_{0} + \underbrace{\frac{2e^{-sN}}{s^3}}_{0} + \frac{2}{s^3} = \frac{2}{s^3}$

prove that $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{t^2\} = \frac{2}{s^3}$, $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$.

$\mathcal{L}\{t\} = \frac{1}{s^2}$

using The definition of The transform we compute

$$\mathcal{L}\{f\}_{(s)} = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{N\to\infty} \int_0^N e^{-st}\cdot t\,dt$$

$u = t \rightarrow du = dt$

$dv = e^{-st}\,dt \rightarrow v = \frac{e^{-st}}{-s}$ $= \lim_{N\to\infty}\left(t\frac{e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\,dt\right)\Big|_0^N$

$$= \lim_{N\to\infty} \frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2}\Big|_0^N = \lim_{N\to\infty}\left(\frac{Ne^{-sN}}{-s} - \frac{e^{-sN}}{s^2}\right) + \frac{1}{s^2}$$

$$= \frac{1}{s^2}$$

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \int_0^\infty e^{-st} t^3\,dt = t^3\frac{e^{-st}}{-s} + 3\int t^2 \frac{e^{-st}}{s}\,dt$$

$u = t^3 \Rightarrow du = 3t^2\,dt$

$dv = e^{-st}\,dt \Rightarrow v = \frac{e^{-st}}{-s}$ $= t^3\frac{e^{-st}}{-s} - 3t^2\frac{e^{-st}}{s^2} - 6\int t\frac{e^{-st}}{s^2}\,dt$

$u = t^2 \rightarrow du = 2t\,dt$

$dv = \frac{e^{-st}}{s}\,dt \rightarrow v = \frac{e^{-st}}{-s^2}$ $= t^3\frac{e^{-st}}{-s} - 3t^2\frac{e^{-st}}{s^2} + 6t\frac{e^{-st}}{s^3} - \int 6\frac{e^{-st}}{s^3}\,dt\Big|_0^N$

$u = t \rightarrow du = dt$

$dv = \frac{e^{-st}}{s^2}\,dt \rightarrow v = \frac{-e^{-st}}{s^3}$

$$= N^3\frac{e^{-sN}}{-s} - 3N^2\frac{e^{-sN}}{s^2} + 6N\frac{e^{-sN}}{s^3} + 6\frac{e^{-sN}}{s^4}$$

$$+ 6\frac{1}{s^4} = \boxed{\frac{6}{s^4}}$$

Laplace Transform of the Derivative

Theorem [4] Let $f(t)$ be continuous on $[0,\infty)$ and $f'(t)$ be a piecewise continuous on $[0,\infty)$ with both of exponential order α then, for $s>\alpha$ $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

proof. Since $\mathcal{L}\{f'\}$ exists, we can use integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$ to obtain

$$\mathcal{L}\{f'\}(s) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv} = \lim_{N\to\infty}\int_0^N e^{-st} f'(t)\,dt$$

$$= \lim_{N\to\infty}\left[e^{-st}f(t)\Big|_0^N + s\int_0^N e^{-st}f(t)\,dt\right]$$

$$= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\lim_{N\to\infty}\int_0^N e^{-st}f(t)\,dt.$$

$$= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\,\mathcal{L}\{f\}(s).$$

To evaluate $\lim_{N\to\infty} e^{-sN}f(N)$ we observe that since $f(t)$ is of exponential order α, There exists a constant M such that for N Large, $|e^{-sN}f(N)| \le e^{-sN}Me^{\alpha N} = Me^{-(s-\alpha)N}$

$$\lim_{N\to\infty}|e^{-sN}f(N)| \le \lim_{N\to\infty} Me^{-(s-\alpha)N}. \quad \text{for } s>\alpha.$$

$$= 0.$$

$$= s\mathcal{L}\{f\}(s) - f(0)$$

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\}(s) - f'(0).$$

$$= s\left[s\mathcal{L}\{f\}_{(s)} - f(0) \right] - f'(0)$$

$$\mathcal{L}\{f''\} = s^2 \mathcal{L}\{f\}_{(s)} - s f(0) - f'(0).$$

$$\mathcal{L}\{f'''\} = s\mathcal{L}\{f''\} - f''(0)$$

بدل تكمي $= s$.

using induction we can extend The last Theorem to higher order derivatives of $f(t)$.

Laplace Transform of higher order derivatives:

Theorem 5

Let $f(t), \ldots, f^{(n-1)}(t)$ be continuous in $[0, \infty)$ and $f^{(n)}(t)$ be piecewise continous on $[0, \infty)$ with all These functions of exponential order α. Then for $s > \alpha$,

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$$\mathcal{L}\{f^{(n)}(t)\}_{(s)} = s^n \mathcal{L}\{f\}_{(s)} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Example:

Using The fact That $\mathcal{L}\{\sin bt\}_{(s)} = \frac{b}{s^2+b^2}$ determine $\mathcal{L}\{\cos bt\}$

solution:

Let $f(t) = \cos bt$

$f(0) = 1$

$f'(t) = -b\sin bt$

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)$$

$$\mathcal{L}\{-b\sin bt\} = s\,\mathcal{L}\{\cos bt\} - 1.$$

$$s\,\mathcal{L}\{\cos bt\} = 1 - b\,\mathcal{L}\{\sin bt\}$$

$$\mathcal{L}\{\cos bt\} = \frac{1}{s}\left[1 - b\left(\frac{b}{s^2+b^2}\right)\right]$$

$$= \frac{1}{s}\left[1 - \frac{b^2}{s^2+b^2}\right].$$

$$\mathcal{L}\{\cos bt\} = \frac{1}{s}\left[\frac{s^2}{s^2+b^2}\right] = \frac{s}{s^2+b^2}.$$

Derivatives of The Laplace Transform.

Theorem 6

:- Let $F(s) = \mathcal{L}\{f\}(s)$ and assume that $f(t)$ is a piecewise continuous on $[0,\infty)$ and of exponential order α Then for $s > \alpha$.

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$$

هذا [illegible]
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proof:-

consider the identity

$$\frac{dF}{ds}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right) f(t)\,dt.$$

$$= -\int_0^\infty e^{-st}\,t\,f(t)\,dt = -\mathcal{L}\{t f(t)\}(s). \text{ since}$$

$$\int_0^\infty f(t)\,e^{-st} = F$$