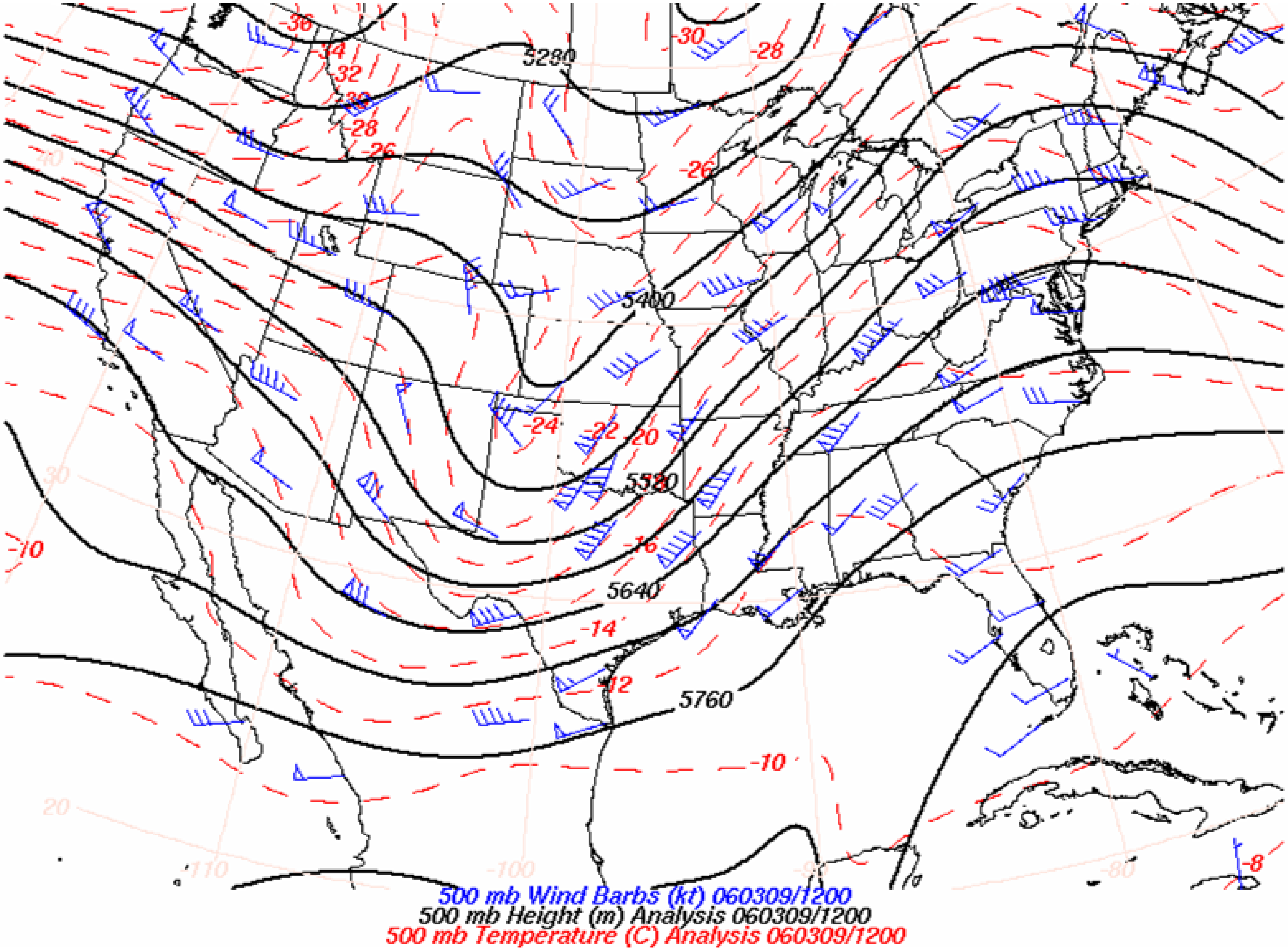
Quasi-Geostrophic (QG) Theory



Quasi-Geostrophic (QG) Theory

QG Theory

- Basic Idea
- Approximations and Validity
- QG Equations / Reference

QG Analysis

- Basic Idea
- Estimating Vertical Motion
    - QG Omega Equation: Basic Form
    - QG Omega Equation: Relation to Jet Streaks
    - QG Omega Equation: Q-vector Form
- Estimating System Evolution
    - QG Height Tendency Equation
- Diabatic and Orographic Processes
- Evolution of Low-level Cyclones
- Evolution of Upper-level Troughs

QG Theory: Basic Idea

Forecast Needs:

- The public desires information regarding temperature, humidity, precipitation, and wind speed and direction up to 7 days in advance across the entire country
- Such information is largely a function of the **evolving synoptic weather patterns** (i.e., surface pressure systems, fronts, and jet streams)

Four Forecast Methods:

Conceptual Models: **Based on numerous observations from past events**
Generalization of the synoptic patterns
Polar-Front theory

Kinematic Approach: **Analyze current observations of wind, temperature, and moisture fields**
Assume clouds and precipitation occur when there is upward motion and an adequate supply of moisture
QG theory

Numerical models: **Based on integration of the primitive equations forward in time**
Require dense observations, and accurate physical parameterizations
User must compensate for erroneous initial conditions and model errors

Statistical models: **Use observations or numerical model output to infer the likelihood of of certain meteorological events**

QG Theory: Basic Idea

What will QG Theory do for us?

- It reveals how ***hydrostatic balance*** and ***geostrophic balance*** constrain and simplify atmospheric motions, but in a realistic manner

- It provides a simple framework within which we can understand and diagnose the ***vertical motion*** and ***evolution*** of three-dimensional synoptic-scale weather systems

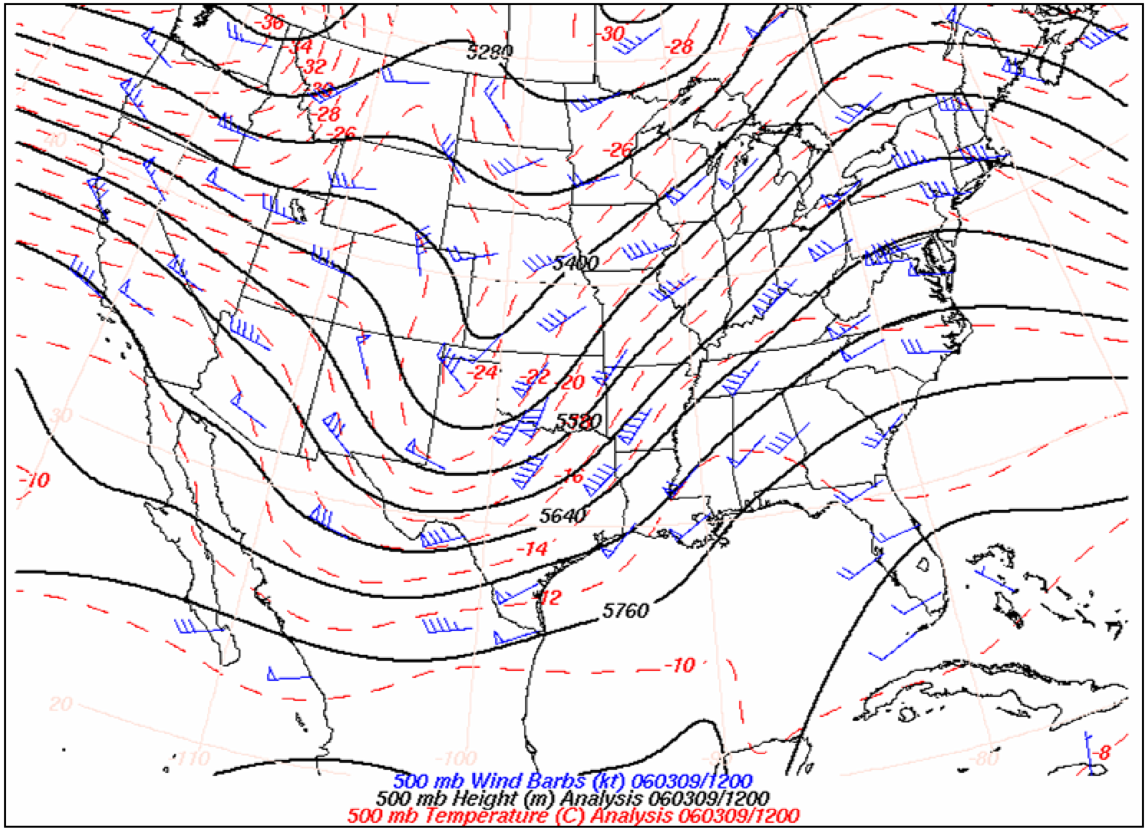
- It helps us to understand how the mass fields (via horizontal temperature advection) and the momentum fields (via horizontal vorticity advection) interact to create vertical circulations that result in ***realistic synoptic-scale weather patterns***

- It offers physical insight into the ***forcing of vertical motion*** and cloud/precipitation patterns associated with mid-latitude cyclones

QG Theory: Approximations and Validity

What do we already know?

- The primitive equations are quite complicated
- For mid-latitude synoptic-scale motions the horizontal winds are nearly geostrophic (i.e., they are **quasi-geostrophic**) above the surface
- We can use this fact to further simplify the equations, and still maintain accuracy



QG Theory: Approximations and Validity

- **Start with**:
  - Primitive equations in isobaric coordinates (to simplify the dynamics)
  - **Hydrostatic Balance** (valid for synoptic-scale flow)
  - **Frictionless Flow** (neglect boundary-layer and orographic processes)

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x} + fv$$ **Zonal Momentum**

$$\frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial y} - fu$$ **Meridional Momentum**

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$ **Hydrostatic Approximation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$ **Mass Continuity**

$$\frac{DT}{Dt} = \omega \frac{RT}{pc_p} + \frac{J}{c_p}$$ **Thermodynamic**

$$p = \rho RT$$ **Equation of State**

QG Theory: Approximations and Validity

- Split the total horizontal velocity into **geostrophic** and **ageostrophic** components

$$u = u_g + u_{ag} \qquad v = v_g + v_{ag}$$

(u_g, v_g) → geostrophic → portion of the total wind in geostrophic balance
(u_{ag}, v_{ag}) → ageostrophic → portion of the total wind NOT in geostrophic balance

• Recall the horizontal equations of motion (isobaric coordinates):

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x} + fv \qquad \frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial y} - fu$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$ and $$\Phi \equiv g\,z$$ $$f \equiv 2\Omega \sin\phi$$

QG Theory: Approximations and Validity

- Perform a **scale analysis** of the acceleration and Coriolis terms (construct ratios):

$$\frac{Du/Dt}{fv} \qquad \frac{Dv/Dt}{fu}$$

- For typical mid-latitude synoptic-scale systems:

$$\frac{Du/Dt}{fv} = \frac{(10\text{ms}^{-1})/10^{5}\text{s}}{(10^{-4}\text{s}^{-1})(10\text{ms}^{-1})} \sim 0.1$$

- Our scale analysis implies a **small** **Rossby Number (R_o):**

$$R_o = \frac{U}{f\,L}$$

where

$$\frac{Du/Dt}{fv} \approx \frac{U}{T}\frac{T}{f\,L} \approx \frac{U}{f\,L}$$

$$R_o \sim 0.1$$

QG Theory: Approximations and Validity

- **Thus**, we can assume:

$$\frac{Du}{Dt} << fv \longrightarrow 0 \approx -\frac{\partial \Phi}{\partial x} + fv \longrightarrow v \approx \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

$$\frac{Dv}{Dt} << -fu \qquad 0 \approx -\frac{\partial \Phi}{\partial y} - fu \qquad u \approx -\frac{1}{f}\frac{\partial \Phi}{\partial y}$$

- Since by definition (geostrophic balance)

We can also assume:

$$v_g \equiv \frac{1}{f}\frac{\partial \Phi}{\partial x} \longrightarrow v \approx v_g \longrightarrow v_g >> v_{ag}$$

$$u_g \equiv -\frac{1}{f}\frac{\partial \Phi}{\partial y} \qquad u \approx u_g \qquad u_g >> u_{ag}$$

- If the ageostrophic component of the wind is small then, we can assume:

$$\frac{Du}{Dt} \approx \frac{Du_g}{Dt_g} \qquad \frac{Dv}{Dt} \approx \frac{Dv_g}{Dt_g}$$

where:

$$\frac{D}{Dt_g} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

Note: This does NOT mean the ageostrophic components are unimportant

QG Theory: Approximations and Validity

- Thus, the **primary assumption** (or simplification) of QG theory is:

R_o is small → (a) ageostrophic flow is assumed to be < 10% of geostrophic flow
(b) horizontal advection is accomplished by only the geostrophic flow
(c) no vertical advection in the total derivative

$$\frac{Du}{Dt} \approx \frac{Du_g}{Dt_g}$$

$$\frac{Dv}{Dt} \approx \frac{Dv_g}{Dt_g}$$

$$\frac{D}{Dt_g} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

- What do our "new" equations of motion look like?

$$\frac{Du_g}{Dt_g} = -\frac{\partial \Phi}{\partial x} + fv$$

$$\frac{Dv_g}{Dt_g} = -\frac{\partial \Phi}{\partial y} - fu$$

- **What do we do with the Coriolis accelerations?**

QG Theory: Approximations and Validity

- We can make **another assumption** about the **Coriolis parameter (f)** that will ultimately simplify our full system of equations:
    - Approximate the Coriolis parameter with a Taylor Series expansion:

$$f = f_0 + \frac{\partial f}{\partial y} y \quad \rightarrow\rightarrow \quad f = f_0 + \beta y$$

where: f_o is the Coriolis parameter at a <u>constant</u> reference latitude

$\beta = \frac{\partial f}{\partial y}$ is the <u>constant</u> meridional gradient in the Coriolis parameter

- If we perform a scale analysis on the two terms, we find: $f_0 >> \beta y$

and we can **<u>re-write</u>** our **geostrophic balance equations** as:

$$v_g \equiv \frac{1}{f}\frac{\partial \Phi}{\partial x} \quad \rightarrow\rightarrow \quad v_g \equiv \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

$$u_g \equiv -\frac{1}{f}\frac{\partial \Phi}{\partial y} \quad \rightarrow\rightarrow \quad u_g \equiv -\frac{1}{f_0}\frac{\partial \Phi}{\partial y}$$

QG Theory: Equations of Motion

- If we now combine (a) our expanded Coriolis parameter with (b) our new geostrophic balance equations, and apply the small Rossby number assumption, we obtain the “final” QG momentum equations:

$$\frac{Du_g}{Dt_g} = f_o v_{ag} + \beta y v_g$$

$$\frac{Dv_g}{Dt_g} = -f_o u_{ag} - \beta y u_g$$

These are (2.14) and (2.15) in the Lackmann text

Physical Interpretation:

(1) Accelerations in the geostrophic flow result entirely from ageostrophic flow associated with the Coriolis force

(2) The Coriolis “torque” acting at right angles to the ageostrophic wind lead to accelerations in the geostrophic wind components perpendicular to the ageostrophic motions

QG Equations: Continuity Equation

- **Start with** the primitive form of the mass continuity equation in isobaric coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

- Substitute in: $v = v_g + v_{ag}$, $u = u_g + u_{ag}$ and then using: $v_g \equiv \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$ $u_g \equiv -\frac{1}{f_0}\frac{\partial \Phi}{\partial y}$

- One can easily show that the **geostrophic flow is non-divergent**, or

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$

- Thus, the QG continuity is:

$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

This is (2.17) in the Lackmann text

Physical Interpretation: The vertical velocity (ω) depends only on the ageostrophic components of the flow

QG Equations: Thermodynamic Equation

• **Start with** the primitive form of the thermodynamic equation in isobaric coordinates:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p} = \omega\frac{RT}{pc_p} + \frac{J}{c_p}$$

• We can combine the two terms containing vertical motion (ω) and apply the primary assumption of QG theory, such that

$$\frac{DT}{Dt_g} = \omega\sigma\frac{p}{R} + \frac{J}{c_p}$$

This is (2.18) in the Lackmann text

where: $$\sigma = -\frac{RT}{p\theta}\frac{\partial\theta}{\partial p}$$

• Finally, (a) neglect diabatic heating (J) [for now…we will return to this later]
(b) assume static stability (σ) is only a function of pressure

$$\frac{DT}{Dt_g} = \omega\sigma\frac{p}{R}$$

Physical Interpretation: (a) geostrophic flow is adiabatic (no latent heating)
(b) horizontally-uniform static stability (CAPE is the same)

QG Equations: Vorticity Equation

- **Start with** the "final" QG momentum equations:

$$\frac{Du_g}{Dt_g} = f_o v_{ag} + \beta y v_g$$ Zonal Momentum

$$\frac{Dv_g}{Dt_g} = -f_o u_{ag} - \beta y u_g$$ Meridional Momentum

- Take $\frac{\partial}{\partial x}$ of the meridional equation and <u>subtract</u> the $\frac{\partial}{\partial y}$ of the zonal equation:

- Then, after some algebra, invoking non-divergent geostrophic flow, and substituting the QG continuity equation, we get:

$$\frac{D\zeta_g}{Dt_g} = -\beta v_g + f_0 \frac{\partial \omega}{\partial p}$$ where: $$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_o}\nabla^2 \Phi$$

Physical Interpretation: The total rate change of geostrophic relative vorticity is a function of (a) the geostrophic advection of planetary vorticity and (b) vortex stretching

QG Equations: Vorticity Equation

- Let's look at the physical interpretations in greater detail:

$$\underbrace{\frac{\partial \zeta_g}{\partial t}}_{\text{Term 1}} = \underbrace{-u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y}}_{\text{Term 2}} \underbrace{-\beta v_g}_{\text{Term 3}} \underbrace{+ f_0 \frac{\partial \omega}{\partial p}}_{\text{Term 4}}$$

Term 1: Local time rate of change in geostrophic relative vorticity

Term 2: Horizontal advection of geostrophic relative vorticity by the geostrophic flow

Positive vorticity advection (PVA) will increase the local vorticity
Negative vorticity advection (NVA) will decrease the local vorticity

Term 3: Meridional advection of planetary vorticity by the geostrophic flow

Beta (β) is always positive
Positive (or northward) meridional flow will decrease the local vorticity
Negative (or southward) meridional flow will increase the local vorticity

Term 4: Addition (subtraction) of vorticity due to stretching (shrinking) of the column

An increase in the vertical motion with height will increase the local vorticity
A decrease in vertical motion with height will decrease the local vorticity

QG Equations: Reference

Summary of the QG Equations:

$$u = u_g + u_{ag} \qquad v = v_g + v_{ag}$$

Decomposition of Total Winds

$$u_g \equiv -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \qquad v_g \equiv \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

Geostrophic Balance

$$f = f_0 + \beta y$$

Coriolis Approximation

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial t} = f_o v_{ag} + \beta y v_g$$

Zonal Momentum Equation

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial t} = -f_o u_{ag} - \beta y u_g$$

Meridional Momentum Equation

$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Continuity Equation

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$

Geostrophic Non-divergence

$$\frac{\partial T}{\partial t} + u_g \frac{\partial T}{\partial x} + v_g \frac{\partial T}{\partial y} = \omega \sigma \frac{p}{R}$$

Adiabatic Thermodynamic Equation

$$\sigma = -\frac{RT}{p\theta}\frac{\partial \theta}{\partial p}$$

Static Stability

$$\frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y} = -\beta v_g + f_0 \frac{\partial \omega}{\partial p}$$

Vorticity Equation

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

Relative Vorticity

QG Theory: Summary

QG Theory – Underlying (Limiting) Assumptions

- Small Rossby Number (advection by only the geostrophic flow)
- Hydrostatic Balance
- Frictionless flow
- No orographic effects ******
- No diabatic heating or cooling ******
- Horizontally uniform static stability

****** We will discuss how to compensate for these two limitations as we progress through each topic

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