

Classical Statistics of Maxwell-Boltzmann

In this section, we review the basic assumptions on which Maxwell-Boltzmann built his statistical distribution. The statistical distribution is nothing but an equation by which the number of n_i particles occupying each of the ϵ_i energy levels in the system is defined, with some limitations. We will find from this distribution all thermodynamic state functions.

The importance of the traditional Maxwell-Boltzmann statistics appears in many important theoretical applications such as calculating the physical properties of an ideal gas. The traditional statistic is only an approximation of advanced statistics such as the Bose-Einstein statistic or the Fermi-Dirac statistic at high temperatures.

1. Hypotheses of classical Maxwell-Boltzmann statistics

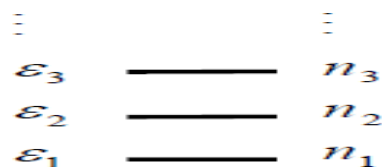
We will assume a system with a large number of particles:

1. Identical
2. Distinguishable
3. To which the Pauli Exclusion Principle does not apply.

Identical particles mean that the particles are identical in composition and components (mass, charge...)

Distinctive particles mean that we can distinguish (or know the difference) between one particle and another even though they match. An example is the coin model where we marked the coins by date or color.

We will assume here that the traditional system (as shown in the figure):



1. It consists of energy levels (ϵ_1 , ϵ_2 , ϵ_3 ,) so that ($\epsilon_1 < \epsilon_2 < \epsilon_3$ )
2. These levels are saturated with a number of particles (n_1 , n_2 , n_3 ,)

Subject to the following two conditions:

1. The total number of N particles in the system is a fixed number, and it can be written mathematically as:

$$N = \sum_i n_i, \quad (1)$$

2. The total internal energy U of the particles in the system is a constant quantity, and it is defined mathematically as:

$$U = \sum_i n_i \varepsilon_i \quad (2)$$

The number of ways needed to distribute N number of characteristic particles over j number of levels so that n_1 is placed in the first level and n_2 in the second level and so is:

$$\tilde{\omega} = \omega\{n_i\} = \frac{N!}{\prod_{i=1}^j n_i!} \quad (3)$$

When a system consisting of an infinite number of particles and levels reaches the state of equilibrium, i.e. the highest possible distribution that contains a set of numbers n_i , the thermodynamic probability value ω reaches a maximum value. Therefore, any change, even a slight, in the set of numbers $\{n_i\}$ will not affect the value of ω .

This means that the relationship:

$$\delta \tilde{\omega} = 0$$

Realized under change:

$$\tilde{n}_i \rightarrow \tilde{n}_i + \delta n_i$$

Subject to the following two conditions:

$$\delta N = \sum_i \delta n_i = 0 \quad (4)$$

$$\delta U = \sum_i \varepsilon_i \delta n_i = 0 \quad (6)$$

The equivalent condition can be used:

$$\delta \ln \tilde{\omega} = 0$$

Instead of using:

$$\delta \tilde{\omega} = 0$$

This is because the logarithmic function is a monotonic function and is easy to work with.

And since the value of ω reaches astronomical numbers, but the value of $(\ln \omega)$ is considered finite and can be dealt with mathematically.

Example 1:

From equation (3), using the Stirling approximation

$$\ln(z!) = z \ln(z) - z$$

Prove that:

$$\ln(\tilde{\omega}) = N \ln N - \sum_i n_i \ln n_i \quad (7)$$

And it was proved that:

$$\delta \ln(\tilde{\omega}) = - \sum_i \ln n_i \delta n_i \quad (8)$$

Using the logarithm of equation (3), we find that:

$$\begin{aligned}
\ln(\tilde{\omega}) &= \ln\left(\frac{N!}{\prod_{i=1}^n n_i!}\right) = \ln(N!) - \ln\left(\prod_{i=1}^n n_i!\right) \\
&= \ln(N!) - \ln[n_1! \times n_2! \times \dots \times n_n!] \\
&= \ln(N!) - [\ln(n_1!) + \ln(n_2!) + \dots] \\
&= \ln(N!) - \sum_i \ln(n_i!)
\end{aligned}$$

Using Stirling's approximation to the previous equation, we find that:

$$\begin{aligned}
\ln(\tilde{\omega}) &= N \ln N - N - \sum_i (n_i \ln n_i - n_i) \\
&= N \ln N - N - \sum_i n_i \ln n_i - \underbrace{\sum_i n_i}_N \\
&= N \ln N - \sum_i n_i \ln n_i
\end{aligned} \tag{7a}$$

It needs to be proven first.

Differentiating equation (7a) with respect to n_i , we find that:

$$\begin{aligned}
\delta \ln(\tilde{\omega}) &= \delta(N \ln N) - \sum_i \ln n_i \delta n_i - \sum_i n_i \underbrace{\delta(\ln n_i)}_{\frac{1}{n_i} \delta n_i} \\
&= 0 - \sum_i \ln n_i \delta n_i - \underbrace{\sum_i \delta n_i}_{=0} \\
&= -\sum_i \ln n_i \delta n_i
\end{aligned} \tag{8a}$$

This needs to be proven again.

Equation (8a) gives the change in the value of $\ln(\omega)$ with the change in the number of particles $\delta(n_i)$ for each level.

**** To calculate the statistical distribution with conditions 1 and 2 and the thermodynamic probability (Equation 3), we take the following two cases :**

1. Non-Degenerate levels (وحيدة الانتماء) المستويات المنفردة

Due to the constraints applied to the system, the change of n_i will not become independent. Therefore, we will use Lagrange's method, by multiplying equation (1) by the coefficient (α -) and equation (2) by the coefficient (β) and adding them to equation (8), we find that:

$$\delta \left[\ln(\tilde{\omega}) + \alpha \sum_i n_i - \beta \sum_i \varepsilon_i n_i \right] = 0 \quad (11)$$

α and β are called "Lagrangian" coefficients, and they are related to the physical properties of the system.

And from the differential equation 11:

$$\sum_i [-\ln n_i + \alpha - \beta \varepsilon_i] \delta n_i = 0$$

It represents an infinite number of equations.

Since the change $\delta(n_i)$ is optional and independent, so by equating the coefficients of $\delta(n_i)$ to zero, we arrive at the most famous relationship for the statistical distribution, which is:

$$\boxed{n_i = e^{\alpha - \beta \varepsilon_i}} \quad (12)$$

Classical Maxwell-Boltzmann Statistical Distribution

وتسمى المعادلة (١٢) بالتوزيع الإحصائي التقليدي لماكسويل -بولتزمان.

And the quantity:

$$e^{-\beta \epsilon_i}$$

It's called the Boltzmann coefficient, and it's a function that depends on temperature and volume.

2. Degenerate States *المستويات متعددة الانتماء*

If we assume that each energy level i is degenerate states, such that it consists of g_i number of energy levels, then each n_i particle can be stabilized in any level in a number of ways:

$$g_i^{n_i}$$

For all energy levels, we find that the number of methods takes the final form:

$$\prod_{i=1}^j g_i^{n_i}$$

From it we find that the number of microscopic levels of the characteristic (distinguishes) particles is known by the equation:

$$\tilde{\omega}\{n_i\} = N! \prod_{i=1}^j \frac{g_i^{n_i}}{n_i!} \quad (13)$$

And if the particles are undistinguished, then we must divide by the value N to get:

$$\tilde{\omega}\{n_i\} = \frac{N!}{N!} \prod_{i=1}^j \frac{g_i^{n_i}}{n_i!} = \prod_{i=1}^j \frac{g_i^{n_i}}{n_i!} \quad (13a)$$

Homework 1:

1. From equation 13 prove that:

$$\ln(\tilde{\omega}) = \sum_i n_i \left(\ln \frac{g_i}{n_i} + 1 \right)$$

2. And it was proved that:

$$\delta \ln(\tilde{\omega}) = \sum_i \ln \frac{g_i}{n_i} \delta n_i \quad (14)$$

Using equations (1), (2) and (14) and Lagrange coefficients α and β , prove that the classical Maxwell-Boltzmann statistical distribution takes the form:

$$\boxed{n_i = g_i e^{\alpha - \beta \varepsilon_i}} \quad (15)$$

The applications of Maxwell-Boltzmann statistics, through equations (12) and (15), are many and varied.

One of the most important special applications of this distribution is the study of the process of light amplification by studying the interaction of materials with an external field of energy, such as electromagnetic radiation (Interaction of radiation with matter), the effect of which appears through photons on the electrons of materials.

There are some important relationships :

1. Calculate the coefficient e^α

And it is calculated using the prerequisite:

$$N = \sum_i n_i = \sum_i g_i e^{-\alpha - \beta \varepsilon_i} = e^{-\alpha} \sum_i g_i e^{-\beta \varepsilon_i}$$

From it we find that:

$$e^{-\alpha} = \frac{N}{\sum_i g_i e^{-\beta \epsilon_i}} = \frac{N}{Z_{sp}}, \quad Z_{sp} = \sum_i g_i e^{-\beta \epsilon_i}$$

The new Z_{sp} function of a single particle, which is known as the Partition function, was originally derived from the German term "Zustandsumme" which means the summation of states (levels). So we'll use the German translation and call it a "level partition function", or for short a partition function.

Using the partition function takes the Maxwell-Boltzmann distribution, equation (15)

$$n_i = N \frac{e^{-\beta \epsilon_i}}{Z_{sp}}; \quad (16)$$

or the general form:

$$n_i = N g_i \frac{e^{-\beta \epsilon_i}}{Z_{sp}} \quad (17)$$

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$$\begin{aligned} \frac{\partial Z_{sp}}{\partial \beta} &= \frac{\partial}{\partial \beta} \sum_i g_i e^{-\beta \epsilon_i} = -\sum_i \epsilon_i g_i e^{-\beta \epsilon_i} \\ \Rightarrow \boxed{\frac{\partial Z_{sp}}{\partial \beta} &= -\sum_i \epsilon_i g_i e^{-\beta \epsilon_i}} \end{aligned} \quad (18)$$

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$$\begin{aligned} \left(\frac{\partial Z_{sp}}{\partial V} \right)_{T,N} &= \left(\frac{\partial}{\partial V} \sum_i g_i e^{-\beta \epsilon_i} \right)_{T,N} = -\sum_i \beta g_i e^{-\beta \epsilon_i} \left(\frac{\partial \epsilon_i}{\partial V} \right)_{T,N} \\ \Rightarrow \boxed{\left(\frac{\partial Z_{sp}}{\partial V} \right)_{T,N} &= -\beta \sum_i g_i e^{-\beta \epsilon_i} \left(\frac{\partial \epsilon_i}{\partial V} \right)_{T,N}} \end{aligned} \quad (19)$$

4. Average Energy Calculation:

Calculate the mean energy from the relationship:

$$\begin{aligned}\bar{E} &= \frac{1}{N} \sum_i n_i \varepsilon_i = \frac{1}{Z_{\text{sp}}} \sum_i \varepsilon_i g_i e^{-\beta \varepsilon_i} \\ &= -\frac{1}{Z_{\text{sp}}} \frac{\partial}{\partial \beta} \sum_i g_i e^{-\beta \varepsilon_i} = -\frac{1}{Z_{\text{sp}}} \frac{\partial Z_{\text{sp}}}{\partial \beta} \\ \Rightarrow \quad &\boxed{\bar{E} = -\frac{\partial}{\partial \beta} [\ln(Z_{\text{sp}})]} \quad (20)\end{aligned}$$

And the total internal energy becomes:

$$\boxed{U = -N \frac{\partial}{\partial \beta} [\ln(Z_{\text{sp}})]} \quad (21)$$

Homework 2:

1. Prove that:

$$\overline{E^2} = \frac{1}{Z_{\text{sp}}} \frac{\partial^2 Z_{\text{sp}}}{\partial \beta^2}$$

2. Prove that the standard deviation is given by:

$$\sigma = \sqrt{\overline{E^2} - (\bar{E})^2} = \sqrt{\frac{\partial^2}{\partial \beta^2} \ln Z_{\text{sp}}}$$

5. Calculation of the average pressure:

The mean pressure is calculated in thermodynamics by:

$$\begin{aligned}
\bar{P} &= -\left(\frac{\partial \bar{U}}{\partial V}\right)_{S,N} \\
&= \frac{1}{Z_{\text{sp}}} \sum_i g_i e^{-\beta \varepsilon_i} \left(-\frac{\partial \varepsilon_i}{\partial V}\right)_{T,N} \\
&= \frac{1}{\beta Z_{\text{sp}}} \left(\frac{\partial Z_{\text{sp}}}{\partial V}\right) = \frac{1}{\beta} \left(\frac{\partial \ln(Z_{\text{sp}})}{\partial V}\right)_{T,N} \\
\Rightarrow \quad &\boxed{\bar{P} = \frac{1}{\beta} \left(\frac{\partial \ln(Z_{\text{sp}})}{\partial V}\right)_{T,N}} \quad (22)
\end{aligned}$$

6. S entropy calculation

From equation (16) and using the Sterling approximation, we find that:

$$\ln(n_i) = \ln N - \ln Z_{\text{sp}} - \beta \varepsilon_i$$

From equation (7), the entropy is calculated as follows:

$$S = k_B \ln(\tilde{\omega}) = k_B \left(N \ln N - \sum_i n_i \ln n_i \right)$$

From there we get:

$$\begin{aligned}
S &= k_B \left(N \ln N - \sum_i n_i \underbrace{\ln n_i}_{\ln N - \ln Z_{\text{sp}} - \beta \varepsilon_i} \right) \\
&= k_B \left(N \ln N - \ln N \underbrace{\sum_i n_i}_N + \ln Z_{\text{sp}} \underbrace{\sum_i n_i}_N + \beta \underbrace{\sum_i n_i \varepsilon_i}_U \right) \\
\Rightarrow \quad &\boxed{S = k_B N \ln Z_{\text{sp}} + k_B \beta U} \quad (23)
\end{aligned}$$

Where we used relations:

$$\sum_i n_i = N, \quad \text{and} \quad \sum_i n_i \varepsilon_i = U$$

7. Calculation of the coefficient β

The coefficient β is calculated as:

Using the law of thermodynamics:

$$TdS = dU + PdV$$

We find that:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)_V \quad (a)$$

From the entropy differential in equation (23) we get:

$$\begin{aligned} \left(\frac{\partial S}{\partial U} \right)_V &= k_B N \underbrace{\frac{d \ln Z_{sp}}{d \beta}}_{-\frac{U}{N} \text{ from (21)}} \left(\frac{\partial \beta}{\partial U} \right)_V + k_B \beta + k_B U \left(\frac{\partial \beta}{\partial U} \right)_V \\ &= k_B \beta \end{aligned} \quad (b)$$

By equating equations a and b, we get:

$$\begin{aligned} \frac{1}{T} &= k_B \beta \\ \Rightarrow \quad &\boxed{\beta = \frac{1}{k_B T}} \end{aligned} \quad (23)$$

Homework 3:

The system consists of three energy levels as follows:

$$\epsilon_3 = 200 k_B \quad \text{و} \quad \epsilon_2 = 100 k_B \quad \text{و} \quad \epsilon_1 = 0$$

And their degenerates are:

$$g_3 = 5 \quad \text{و} \quad g_2 = 3 \quad \text{و} \quad g_1 = 1$$

Calculate the relative population between levels at temperature

$$T = 100 \text{ K.}$$

Note:

From equation (15), we find that the relative population between levels is given by the relationship:

$$\frac{n_j}{n_i} = \frac{g_j e^{\alpha - \beta \epsilon_j}}{g_i e^{\alpha - \beta \epsilon_i}} = \frac{g_j e^{-\beta \epsilon_j}}{g_i e^{-\beta \epsilon_i}}$$

And from there we find that:

$$\begin{aligned} \frac{n_2}{n_1} &= \frac{3e^{-100 k_B / 100 k_B}}{1e^{-0}} = 3e^{-1} \\ \frac{n_3}{n_1} &= \frac{5e^{-200 k_B / 100 k_B}}{1e^{-0}} = 5e^{-2} \\ \frac{n_3}{n_2} &= \frac{5e^{-200 k_B / 100 k_B}}{3e^{-100 k_B / 100 k_B}} \approx 1.7e^{-1} \end{aligned}$$

Partition Function for Non-Degenerate States

دالة التجميع للمستويات المنفردة

It is the most important function in statistical physics, which is the Z_{sp} levels partition function, which is considered the cornerstone in the calculations of thermodynamic properties and functions.

The Z_{sp} function is a partition of the Boltzmann coefficient $e^{-\beta \epsilon}$ and has the following properties :

1. The partition of all the distinct microscopic levels of the system as a whole, i.e.

$$Z_{sp} = \sum_i e^{-\beta \epsilon_i} \quad (1)$$

2. Or it is a summation of all energy levels characteristic of the system as a whole, meaning that:

$$Z_{sp} = \sum_i g_i e^{-\beta \epsilon_i} \quad (2)$$

3. It depends on the properties of the system and its statistical microstructure.

4. Depends on the absolute temperature of the system by the coefficient β

5. It is proportional to the volume of the gas through the density of the levels $g(\mathcal{E})$ or g_i .

6. Its numerical value changes from value 1 to astronomical values.

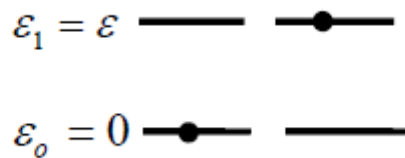
Example 1:

Consider a simple two-particle system. If the system exists in only two energy levels:

$$. g_1=g_2=1 \quad \text{مع الشرط} \quad \mathcal{E}_2 = \epsilon \quad \text{و} \quad \mathcal{E}_1 = 0$$

1. Calculate Z_{sp}
2. Calculate Z_2 if the two particles are distinct (Boltzmann particles).
3. Calculate Z_2 if the two particles are indistinguishable.

1. For a single particle, we find that the number of microscopic states is two, as follows:

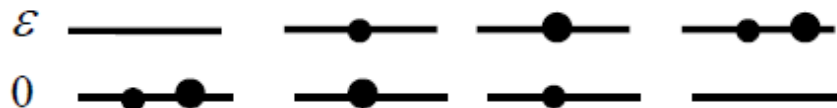


So we find that the partition function is:

$$Z_{sp} = \sum_{i=1}^2 e^{-\beta \epsilon_i} = e^{-\beta \epsilon_1} + e^{-\beta \epsilon_2}$$

$$= e^0 + e^{-\beta \epsilon} = 1 + e^{-\beta \epsilon}$$

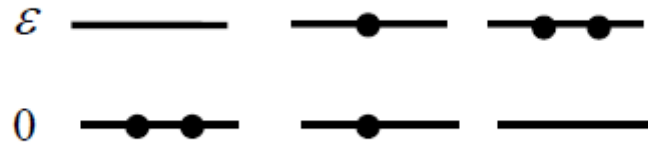
2. In the event that the two particles are distinct, we find that the number of microscopic states is four levels:



The partition function is calculated by the equation:

$$\begin{aligned}
Z_2 &= \sum_{i=1}^4 e^{-\beta \varepsilon_i} = e^{-\beta \varepsilon_1} + e^{-\beta \varepsilon_2} + e^{-\beta \varepsilon_3} + e^{-\beta \varepsilon_4} \\
&= e^{-\beta(0+0)} + e^{-\beta(0+\varepsilon)} + e^{-\beta(\varepsilon+0)} + e^{\beta(-\varepsilon-\varepsilon)} \\
&= 1 + 2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} = (1 + e^{-\beta\varepsilon})^2 \\
&= (Z_{sp})^2
\end{aligned}$$

3. In the case that the two particles are indistinguishable, we find that there are only three microscopic levels.



The partition function is calculated by the equation:

$$\begin{aligned}
Z_2^{\setminus} &= \sum_{i=1}^4 e^{-\beta \varepsilon_i} = e^{-\beta \varepsilon_1} + e^{-\beta \varepsilon_2} + e^{-\beta \varepsilon_3} + e^{-\beta \varepsilon_4} \\
&= e^{-\beta(0+0)} + e^{-\beta(0+\varepsilon)} + e^{\beta(-\varepsilon-\varepsilon)} \\
&= 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} \\
&\neq (Z_{sp})^2
\end{aligned}$$

Note:

That in the general case of characteristic and identical classical particles, the partition function of each single particle is equal.

Thus, the relationship between the total partition function Z_N and the partition function for single particle Z_{sp} is given by:

$$Z_N = (Z_{sp})^N \quad (3)$$

This does not apply to undistinguishable particles.

وهذا لا ينطبق على الجسيمات الغير المميزة.

We also note that the numerical values of the partition function in the case of distinguishable particles are more than the numerical values of the partition function in the case of undistinguished particles.

In general, for uncharacterized particles (undistinguishable particles), we find that the partition function is calculated from the following equation:

$$\boxed{Z_N = \frac{Z_{sp}^N}{N!}} \quad (4)$$

ويصبح توزيع ماكسويل - بولتزمان بالشكل:

$$n_i = N g_i \frac{e^{-\beta \varepsilon_i}}{Z_{sp}} \quad (5)$$

The Maxwell-Boltzmann distribution function in the form:

$$f(\varepsilon_i) = \frac{n_i}{g_i} = N \frac{e^{-\beta \varepsilon_i}}{Z_{sp}} \quad (6)$$

It specifies the average number of particles for each quantum level.

We can also know the ratio:

$$P_i = \frac{n_i}{N} = g_i \frac{e^{-\beta \varepsilon_i}}{Z_{sp}}, \text{ with } N \rightarrow \infty \quad (7)$$

It is the random probability of having a particle in plane i with the two conditions:

$$\sum_i P_i = 1 \quad \text{and} \quad \sum_i \varepsilon_i P_i = \bar{E}$$

And the mean value of any physical quantity takes the form:

$$\bar{f} = \frac{1}{N} \sum_i n_i f(\varepsilon_i) = \frac{1}{Z_{sp}} \sum_i g_i f(\varepsilon_i) e^{-\beta \varepsilon_i} \quad (8)$$

Homework 4:

Using the definition:

$$P_i = \frac{n_i}{N}$$

Were $N \rightarrow \infty$, prove that

$$s = \frac{S}{N} = -k_B \sum_i P_i \ln P_i \quad (9)$$

Example 2:

The system consists of three energy levels as follows:

$$\epsilon_3 = 200 k_B \quad , \quad \epsilon_2 = 100 k_B \quad , \quad \epsilon_1 = 0$$

And their degenerates are:

$$g_3 = 5 \quad , \quad g_2 = 3 \quad , \quad g_1 = 1$$

Calculate:

1. Partition function Z_{sp}
2. Relative population for each level and average energy at temperature $T = 100$ K.

The partition function is calculated from the equation:

$$\begin{aligned} Z_{sp} &= \sum_i g_i e^{-\beta \epsilon_i} \\ &= 1 + 3e^{-100 k_B / 100 k_B} + 5e^{-200 k_B / 100 k_B} \\ &= 1 + 3e^{-1} + 5e^{-2} = 2.78 \end{aligned}$$

The relative housing for each level is given by the relationship:

$$P_i = \frac{n_i}{N} = g_i \frac{e^{-\beta \epsilon_i}}{Z}$$

Thus, the probability of the presence of the particle at each level is given by the values:

$$P_0 = \frac{1}{Z} = 0.360, \quad P_1 = \frac{3e^{-1}}{Z} = 0.397, \quad P_2 = \frac{5e^{-2}}{Z} = 0.243$$

i	g_i	ε_i	$g_i e^{-\beta \varepsilon_i}$	$P_i = g_i \frac{e^{-\beta \varepsilon_i}}{Z_{sp}}$	$\varepsilon_i P_i$
0	1	0	1	0.360	.
1	3	$100 k_B$	$3e^{-1}$	0.397	$39.7 k_B$
2	5	$200 k_B$	$5e^{-2}$	0.243	$48.6 k_B$
			$Z_{sp} = 2.78$	Total = 1	$\bar{E} = 88.3 k_B$

والطاقة المتوسطة:

$$\bar{E} = \sum_i \varepsilon_i P_i = (0 \times P_0 + 100 \times P_1 + 200 \times P_2) k_B = 88.3 k_B$$

Partition Function for Degenerate States:

When the levels are close to each other (connected), we can dispense with the partition function:

$$Z_{sp} = \sum_i g_i e^{-\beta \varepsilon_i}$$

$$Z_{sp} = \int g(\varepsilon) e^{-\beta \varepsilon} d\varepsilon \quad (1)$$

Where $g(\varepsilon)$ density levels

The general form of the partition function becomes:

$$Z_{sp} = \frac{1}{h^3} \int g(\varepsilon) e^{-\beta \varepsilon} d\varepsilon d\mathbf{r} \quad (2)$$

N is the number of particles in the system, ε is the total energy of the particle, and \mathbf{r} is the space coordinates of the particle.

For connected levels, the Maxwell-Boltzmann distribution function takes the form:

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = N \frac{e^{-\beta \varepsilon}}{\int g(\varepsilon) e^{-\beta \varepsilon} d\varepsilon} \quad (3)$$

Example 3:

Use equation (2) to calculate the partition function for monatomic gases (ideal gas).

*****An ideal gas has only translation kinetic energy (we neglect the interaction energy between atoms).***

Since the gas occupies a large space and the energy levels converge in it, it can be considered that the g_i levels density function is a continuous function. So we can replace it with the quantity:

$$g(\varepsilon)d\varepsilon$$
$$\cdot \varepsilon + d\varepsilon \rightarrow \varepsilon$$

And to calculate the partition function for a single particle Z_{sp} :

From equation (2) where the density of levels is known by the equation:

$$g(\varepsilon) = \frac{V}{2\pi^2} \left(\frac{m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon}$$

$$\begin{aligned} Z_{sp} &= \int_0^{\infty} g(\varepsilon) e^{-\beta\varepsilon} d\varepsilon = \frac{V}{2\pi^2} \left(\frac{m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \sqrt{\varepsilon} e^{-\varepsilon/k_B T} d\varepsilon \\ &= \frac{V}{2\pi^2} \left(\frac{m}{\hbar^2} \right)^{3/2} (k_B T)^{3/2} \underbrace{\int_0^{\infty} \sqrt{y} e^{-y} dy}_{\Gamma(3/2) = \frac{\sqrt{\pi}}{2}} = \frac{V}{2\pi^2} \left(\frac{m}{\hbar^2} \right)^{3/2} (k_B T)^{3/2} \frac{\sqrt{\pi}}{2} \\ &= V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} = V \left(\frac{2\pi m}{\beta h^2} \right)^{3/2} \end{aligned}$$

For the N number of characteristic Maxwell particles, we find that the total partition function is:

$$Z_N = (Z_{sp})^N = V^N \left(\frac{2\pi m}{\beta h^2} \right)^{3N/2} \quad (4)$$

We often use the logarithm of the partition function and it takes the form:

$$\ln(Z_{sp})^N = N \ln V + N \ln \left(\frac{2\pi m}{\beta h^2} \right)^{3/2}, \quad \beta = 1/k_B T \quad (4a)$$

$$= N \ln V + \frac{3}{2} N \ln \left(\frac{2\pi m k_B}{h^2} \right) + \frac{3}{2} N \ln(T) \quad (4b)$$

Some of the applications of the Partition Function of an ideal gas are:

بعض تطبيقات دالة التجميع للغاز المثالي هي:

The properties of an ideal gas are defined by physical quantities such as pressure and internal energy and specific heat. These quantities have been calculated by thermodynamics and kinetic theory.

Here we will use the partition function, equation (4), to calculate these quantities.

1. Pressure calculation

Calculate the ideal gas pressure

Starting from equation (4), we find that the free energy of Helmholtz gives:

$$F = -k_B T \ln(Z_{sp})^N = -N k_B T \ln(Z_{sp})$$

The pressure is calculated from the relationship:

$$\begin{aligned}
p &= -\left(\frac{\partial F}{\partial V}\right)_{T,\beta} = Nk_B T \left(\frac{\partial \ln Z_{sp}}{\partial V}\right)_{T,N} = Nk_B T \frac{\partial \{(4.a)\}}{\partial V} \\
&= Nk_B T \frac{1}{V} \\
&\Rightarrow \boxed{pV = Nk_B T}, \quad (5)
\end{aligned}$$

وهي المعادلة العامة للغاز المثالي.

2. Calculation of internal energy

The internal energy of an ideal gas is calculated from the relationship:

$$\begin{aligned}
U &= -\frac{\partial}{\partial \beta} \ln Z_N = -\frac{\partial}{\partial \beta} \{(13.b)\} \\
&= -\left(\frac{3}{2} \beta \left(-\frac{1}{\beta^2}\right)\right) = \frac{3}{2} \frac{N}{\beta} = \frac{3}{2} Nk_B T \\
&\Rightarrow \boxed{U = \frac{3}{2} Nk_B T} \quad (6)
\end{aligned}$$

From here, we find that the average kinetic energy of a particle is:

$$\boxed{\bar{\varepsilon} = \frac{U}{N} = \frac{3}{2} k_B T} \quad (7)$$

This corresponds to the law of equal distribution of energy over degrees of freedom

3. Calculation of specific heat:

The specific heat of an ideal gas is calculated from the relationship:

$$\boxed{C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{3}{2} Nk_B} \quad (8)$$

And from them we find that it is a fixed quantity and does not depend on temperature.

4. Entropy calculation

The entropy is calculated as:

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = Nk_B \left[\ln V + \frac{3}{2} \ln T + S_0 \right], \quad (9)$$

$$S_0 = \frac{3}{2} \left[\ln \left(\frac{2\pi mk}{h^2} \right) + 1 \right]$$

S_0 كمية ثابتة ولا تعتمد علي درجة الحرارة T او V او N