Q_1: $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = [n,0]$, $g : \mathbb{Z} \to \mathbb{Q}$, $g(m) = [m,1]$

$g \circ f(n) = g([n,0]) = [[n,0],1]$, but $[n,0] = n$

$\therefore g \circ f(n) = [n,1]$.

$(g \circ f)(5) = [5,1]$

T.P. $g \circ f$ is $1-1$.

Let $n, m \in \mathbb{N}$, s.t. $(g \circ f)(n) = (g \circ f)(m)$

$[n,1] = [m,1]$, $n, m \in \mathbb{N}(\mathbb{Z})$.

$\Leftrightarrow n \cdot 1 = m \cdot 1$ Def of $\sim^*$ on $\mathbb{Z} \times \mathbb{Z}$

$\Leftrightarrow n = m$ (prop. of $\cdot$ on $\mathbb{N}(\mathbb{Z})$.

$\therefore g \circ f$ is $1-1$ function.

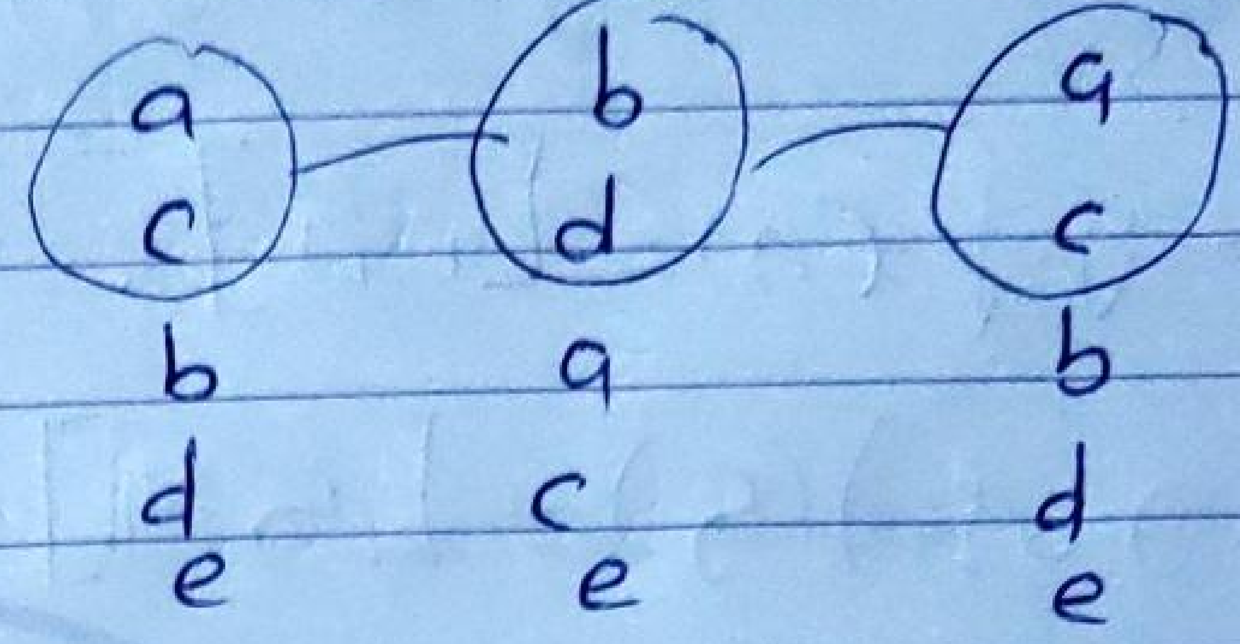
First Attempt

Q_2. (i) $A=\{a,b,c,d,e\}$ and f involution function.

$f(\{a,c\})=\{b,d\}$.

Sol

$f=f^{-1}$ or $f^{-1}of=I \wedge fof^{-1}=I$



$f^{-1}(b)\cup f^{-1}(d)$

$=f^{-1}(\{b,d\})$

but since $f(\{a,c\})=\{b,d\} \rightarrow f^{-1}of(\{a,c\})=f^{-1}(\{b,d\})$

$\rightarrow I(\{a,c\}) = f^{-1}(\{b,d\})$

$\{a,c\}=f^{-1}(\{b,d\})$.

since $f^{-1}(\{b,d\})=\{a,c\} \wedge f=f^{-1}$

$\therefore f(\{b,d\})=\{a,c\}$.

Therefore $f(\{b,d\})\cup f(\{a,c\})=f(\{a,b,c,d\})$

$\downarrow$

$\{a,c\}\cup\{b,d\} = f(\{a,b,c,d\})$

$\rightarrow f(\{a,b,c,d\})=\{a,c,b,d\}$.

So, $f(e)$ must equal to e; that is, $f(e)=e$.

First Attempt

Q_2 (ii) $P_1 : \mathbb{N} \times \mathbb{Z} \to \mathbb{N}$, $f : \mathbb{N} \to \mathbb{Z}$, $f(x) = \sqrt{|x|}$

Sol

$\therefore \; f^{-1}(x) = \{y \in \mathbb{N} / f(y) = x\}$

$= \{y \in \mathbb{N} / \sqrt{|y|} = x\} = \{y \in \mathbb{N} \mid x^2 = |y|\}$

$= \{y \in \mathbb{N} \mid y = x^2\}$ since $y \in \mathbb{N} \to y \geq 0 \to |y| = y$.

$\therefore \; f^{-1}(x) = x^2$.

$P_1^{-1}(x) = \{x\} \times \mathbb{Z}$

$(f \circ P_1)(x,y) = f(P_1(x,y)) = f(x) = \sqrt{|x|}$

$(P_1 \times f)(x,y,z) = P_1(x,y) \times f(z) = (x, \sqrt{|z|})$

$(f \circ P_1)^{-1}(x) = P_1^{-1} \circ f^{-1}(x) = P_1^{-1}(f^{-1}(x)) = P_1^{-1}(x^2) = \{x^2\} \times \mathbb{Z}$.

First Attempt

Q5 (i) $[a,b] \oplus [c,d] = [a+c, b+d]$

$$[a,b] \odot [c,d] = [ac+bd, ad+bc]$$

$12 \odot -2 = [12,0] \odot [0,2] = [12\cdot 0 + 0\cdot 2, 12\cdot 2 + 0\cdot 0]$
$= [0, 24] = -24$

$((-2 \odot 5) \oplus 10) = ([0,2] \odot [5,0]) \oplus [10,0]$

$= [0\cdot 5 + 2\cdot 0, 0\cdot 0 + 2\cdot 5] \oplus [10,0]$
$= [0,10] \oplus [10,0] = [0+10, 10+0] = [10,10] = [0,0]$
$= 0$

(ii) $[2,5] = ([3,6] \odot [5,2]) \odot [a,b]$ in $\mathbb{Q}$.

$[2,5] = [3\cdot 5, 6\cdot 2] \odot [a,b]$ $b \neq 0$
$[2,5] = [15, 12] \odot [a,b] = [15a, 12b]$

$\therefore [2,5] = [15a, 12b]$
$\Leftrightarrow 2\cdot(12b) = 5\cdot(15a)$
$\Leftrightarrow (2\cdot 12)b = (5\cdot 15)a$
$\Leftrightarrow 24b = 75a$
$\Leftrightarrow 3\cdot(8b) = 3(25a) \Leftrightarrow 25a = 8b$ (cancellation law on $\mathbb{Z}$)
$\Leftrightarrow [a,b] = [25,8]$ (Def of L^* on $\mathbb{Z}\times\mathbb{Z}$)

(i) add $((-2 \odot 5) \oplus 10) < 5$.

$([0,2] \odot [5,0]) + [10,0] = [0\cdot 5 + 2\cdot 0, 0\cdot 0 + 2\cdot 5] + [10,0]$
$= [0,10] + [10,0] = [0+10, 10+0] = [10,10] = [0,0]$
$= 0$

Note $((-2 \odot 5) + 10) < 5$ since

$[0,0] < [5,0]$ since $0+0 < 5+0$
$0 < 5$

Q6 (ii)

First Attempt

- $(P(X), \cup)$, ϕ is the identity element since

$\forall A \in P(X)$, $A \cup \phi = \phi \cup A = A$.

- $(P(X), \cap)$, X is the identity element since

$\forall A \in P(X)$, $A \cap X = X \cap A = A$ (since $A \subseteq X$).

- $(\mathbb{Z}, *)$, $a * b = a + b + 1$, $b = -1$ is the identity element

since $a * (-1) = a + (-1) + 1 = a + 0 = a$.

First Attempt

Q7 (i)

$(3+5^+)^+ \cdot 4^+)^+ + 6$

$3+5^+ = (3+5)^+ = 8^+ = 9$

$(3+5^+)^+ = 9^+ = 10$

$(3+5^+)^+ \cdot 4^+ = 10 \cdot 4^+ = 10 + 10 \cdot 4 = 10 + 40$

$= 10 + 39^+ = (10+39)^+ = (49)^+ = 50$

$((3+5^+)^+ \cdot 4^+)^+ = 50^+ = 51$

$((3+5^+)^+ \cdot 4^+)^+ + 6 = 51 + 5^+ = (51+5)^+$

$= 56^+ = 57$

(ii) (a) $\overline{-4} \cdot_{12} (\overline{7} \cdot_{12} \overline{9})$

$\overline{-4} \equiv \overline{8} \mod 12$

$\longrightarrow \overline{8} \cdot_{12} (\overline{7} \cdot_{12} \overline{9}) = \overline{8} \cdot_{12} \overline{63} \equiv \overline{8} \cdot_{12} \overline{3}$

$\equiv \overline{24} \equiv 0 \mod 12$

since $63 = 5 * 12 + 3$

Since $24 = 2 \cdot 12 + 0$

(b) $((23)\circ(123))\circ(123)$

$(23)\circ(123) = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \circ \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&2&3\\2&3&1\\1&3&2\end{pmatrix} = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$

$= (13)$

$\therefore ((23)\circ(123))\circ(123)) = (13)\circ(123) = \begin{pmatrix}1&2&3\\2&3&1\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = (12).$

٧