**Real and Rational Numbers**

1. **Order Sets**
	1. **Definition**: Let $R$ be a relation on a set $X$, we say that
* $R$ is a reflexive on $X$, if $xRx$ $∀$ $x\in X$.
* $R$ is a symmetric on $X$, if $xRy$, then $yRx$.
* $R$ is a transitiveon $X$, if $xRy$, and $yRz$, then $xRz$.
* $R$ is an anti-symmetricon $X$, if $xRy$, and $yRx$, then $x=y$.
	1. **Note**: We say that $R$ is a preorderrelation on $X$, if $R$ is a reflexive and transitive, and $R$ is a partial order relation on $X$, if $R$ is a reflexive, transitive and anti-symmetric.
	2. **Note**: Let $X\ne ∅$, then $R$ is a partial order set, we say that ($X,R$) is a partially order set.
	3. **Example**: The order pairs $(N,\leq )$, $(Z,\leq )$, $(Q,\leq )$, $(R,\leq )$, $(C,\leq )$ where $(\leq =R)$ are a partially order sets.
	4. **Definition**: Let $X$ be a partial order set, and $x,y\in X$ we say $x,y$ are comparable, if $x\leq y$ or $y\leq x$.
	5. **Definition**: Let $A⊆X$, then $A$ is a totally ordered or chain in $X$, if every two elements in $X$ are comparable.
	6. **Example**: The order pairs $(R,\leq )$, $(Q,\leq )$are a totally ordered, but $(C,\leq )$ does not.
	7. **Definition**: Let $X$ be a partial order set and $a,b\in X$, we say that $a$ is a first element or a smallest element in $X$, if $a\leq x$ $∀x\in X$. We say that $b$ is a last element or greatest element in $X$, if $x\leq b$ $∀x\in X$.
	8. **Definition**: Let $X$ be a partial order set, then $X$ is called a well ordered, if every non empty subset of $X$ contains a first element.
	9. **Definition**: Let $X$ be a partial order set and $a,b\in X$, then $a$ is called a minimal element in $X$, if $x\in X$, $x\leq a$, then $a=x$. We say that $b$ is a maximal element in $X$, if $x\in X$, $b\leq x$, then $b=x$.
	10. **Examples**:
1. Let $A=\{-3,-2,-1,0,1,4,7\}$, then max $A=7$ and min$ A=-3$.
2. Min $N=1$ and max $N$ does not exist.
3. Min $Z$ and max $Z$ does not exist.
4. Let $A=\{\frac{1}{n}:n\in Z\}$, then max$ A=1$ and min $A$ does not exist.
5. Let $A=\{-\frac{1}{n}:n\in Z\}$, then min$ A=-1$ and max $A$ does not exist.
6. Let $A=\{\mp \frac{1}{n}:n\in Z\}$, then max$ A=1$ and min $A=-1$.
	1. **Definition**: Let $X$ be a partial order set and $A⊆X$, we say that $a\in X$ be lower bound of $A$, if $a\leq x$ $∀x\in A$. We say that $a$ called a greatest lower bound of $A$, if its:
7. A lower bound of $A$;
8. $a^{'}<a$ for all lower bound $a^{'}$ of $A$.
	1. **Note**: We denote of element which a greatest lower bound of $A$ by inf $A$.
	2. **Definition**: Let $X$ be a partial order set and $A⊆X$, we say that $b\in X$ be upper bound of $A$, if $x\leq b$ $∀x\in A$. We say that $b$ called a smallest upper bound of $A$, if its:
9. An upper bound of $A$;
10. $b<b^{'}$ for all upper bound $b^{'}$ of $A$.
	1. **Note**: We denote of element which a smallest upper bound of $A$ by sup $A$.
	2. **Examples**:
11. Let $A=\{x\in R:x\leq 2\}$, then sup $A=2$ and inf $A$ does not exist.
12. Let $A=\{x\in R:-4\leq x\leq 5\}$, then sup $A=5$ and inf $A=-4$.
	1. **Definition**: Let $X$ be a partial order set and $A⊆X$, we say that $A$ is a bounded below, if there exist a lower bound and $A$ is a bounded above, if there exists an upper bound. We say that $A$ bounded, if $A$ bounded from a lower and an upper.
	2. **Definition**: Let $X$ be a partial order set. We say that $X$ complete or complete ordered, if for all non empty subset and bounded from above $A$ in $X$, then sup $A$ exists.
13. **Real Numbers**

(2.1) **Axioms of Field**

1. **Axioms of abelian.**
* $x+y=y+x ∀x,y\in R$.
* $x.y =y.x ∀x,y\in R$.
1. **Axioms of associative.**
* $x+\left(y+z\right)=\left(x+y\right)+z ∀x,y,z\in R.$
* $x.\left(y.z\right) =\left(x.y\right).z ∀x,y,z\in R$.
1. **Axiom of distribution.**

$$x\left(y+z\right)=xy+xz ∀x,y,z\in R $$

1. **Axioms of identity element.**
* There is $0\in R$ such that $x+0=0+x=x$.
* There is $1\in R$ such that $x.1=1.x=x$.
1. **Axioms of inverse element.**
* For all $x\in R$ there is $-x\in R$ such that $x+\left(-x\right)=\left(-x\right)+x=0$.
* For all $x\in R$, $x\ne 0$ there is $y\in R$ such that $x.y=y.x=1$.

(2.2) **Theorem:** Let $x,y,z\in R, $then

1. $-\left(x-y\right)=y-x$.
2. $x-y=x+(-y)$.
3. $x+z=y+z$ iff $x=y$.
4. If $z\ne 0$, then $xz=yz$ iff $x=y$.
5. $xy=0$ iff $x=0$ or $y=0$.
6. $\left(-x\right)y=x\left(-y\right)=-xy$.
7. $-\left(-x\right)=x$.
8. If $x\ne 0$, then $(-x)^{-1}=-x^{-1}$ and $(x^{-1})^{-1}=x$.

(2.3) **Axioms of order.**

There is a non-empty subset of $R$ which denoted by $R\_{+}$ and its satisfy:

1. If $x,y\in R\_{+}$ then $x+y\in R\_{+}$ and $xy\in R\_{+}$.
2. If $x\in R$ then one of following is true $-x\in R\_{+}$, $x=0$, $x\in R\_{+}$ .

(2.4) **Definition**:

1. If $x,y\in R$ then $x<y$ if $y-x\in R\_{+}$.
2. $x\leq y$ means $x<y$ or $x=y$.
3. $x\leq y<z$ means $y<z$ and $x\leq y$.

(2.5) **Theorem:**

1. For all $x,y\in R$ then either$ x<y$ or $x>y$ or $x=y$.
2. If $x<y$ and $y<z$ then $x<z$.
3. $x+z<y+z$ iff $x<y$.
4. If $x<y$ and $z<w$ then $x+z<y+w$.
5. If $z>0$ then $xz<yz$ iff $x<y$.
6. If $z<0$ then $xz<yz$ iff $x>y$.
7. If $0<x<y$ and $0<z<w$ then $xz<yw$.

(2.6) **The Completeness Axiom.**

Let $∅\ne A⊆R$ then

1. If $A$ is an upper bounded, then sup$ A$ exists.
2. If $A$ is a lower bounded, then inf $A$ exists.

(2.7) **Theorem:** Let $∅\ne A⊆R$ and $a,b\in R$ then

1. Inf $A=a$ iff
2. $a\leq x ∀x\in A$.
3. $∀ε>0 ∃ y\in A \ni y<a+ε$.
4. Sup $A=b$ iff
5. $x\leq b ∀x\in A$.
6. $∀ε>0 ∃ y\in A \ni y>b-ε$.

**Proof:** Let inf $A=a⟹a$ is a lower bound of $A⟹a\leq x ∀x\in A⟹(a)$ satisfies.

Let $ε>0⟹a+ε>a$, since $a$ is greatest lower bound of $A⟹a+ε$ not lower bound of $A⟹∃z\in A\ni z<a+ε⟹(b)$ satisfies.

Now let (a), (b) are satisfy

(a)$ ⟹a$ is a lower bound of $A$, let $c\in R\ni a<c$. We must prove that $c$ not lower of $A$. Put $ε=c-a⟹ε>0⟹∃y\in A\ni y<a+ε⟹y<a+\left(c-a\right)=c⟹$ inf $A=a$

(2) Assume that sup $A=b⟹b$ an upper bound of $A⟹x\leq b ∀x\in A⟹(1)$

Now to prove (2) let $ε>0⟹-ε<0⟹b-ε<b$, since $b$ is a smallest upper bound of $A⟹b-ε$ does not upper bound of $A⟹∃z\in A\ni b-ε<y$.

(1) means $b$ is an upper bound of $A$, let $d\in R\ni d<b$. Put $ε=b-d⟹ε>0⟹∃y\in A$ by (2) $\ni y>b-ε⟹y>b-\left(b-d\right)=d⟹$ sup$ A=b$ $∎$

(2.8) **Theorem:**(Archimedes property)

If $x,y\in R$ and $x>0$ then $∃n\in Z^{+}\ni nx>y$.

**Proof:** Let $∃a,b\in R\ni a>0$ and $na\leq b ∀ n\in N$. Put $A=\{na:n\in N\}$, $1.a=a\in A⟹∅\ne A⊆R$, $na\leq b ∀ n\in N⟹b$ is an upper bound of $A⟹A$ bounded from above. Since$ R$ satisfies the completeness $⟹$ $∃y\in R\ni y=$ sup $A$. $a>0⟹-a<0⟹y-a<y$ since $y$ is a smallest upper bound of $A⟹y-a$ does not upper bound of $A⟹∃m\in N\ni y-a\leq ma⟹y\leq ma+a⟹y\leq \left(m+1\right)a$, since $m+1\in N⟹\left(m+1\right)a\in A⟹y$ does not upper bound of $A⟹$ contradiction $∎$

(2.9) **Corollary**:

1. $∀x\in R\_{+}∃ n\in Z\_{+}\ni \frac{1}{n}<x$.
2. $∀x\in R ∃ n\in Z\_{+} \ni n>x $.
3. $∀x\in R ∃ m,n\in Z \ni m<x<n$.
4. $∀x\in R ∃ $a unique integer $n\in Z \ni n\leq x<n+1$.

**Proof:** (1) Put $b=1$ and $a=ε⟹∃n\in Z\_{+} \ni na>b⟹nε>1⟹\frac{1}{n}<ε$.

(2) Put $b=x$ and $a=1⟹∃n\in Z\_{+} \ni na>b⟹n>x$.

(3) since $x\in R⟹$ by (2) $∃n\in Z\_{+} \ni n>x$, now we must prove that $∃m\in Z\_{+} \ni m<x$. Put $A=\{k\in Z:k>-x\}⟹A⊆R$ and $A$ is a lower bound (since $R$ satisfies completeness) $⟹∃y\in R\ni $ inf $A=y⟹y>-x⟹-y<x$, put $m=-y⟹m<x$.

(4) Put $A=\{m\in Z:m\leq x\}⟹∅\ne A⊆R$ and $A$ has an upper bound, (since $R$ satisfies completeness)$ ⟹∃n\in R\ni $ sup $A=n⟹n\leq x$. To prove$x<n+1$, suppose that $n+1\leq x⟹n+1\in A$, but this is contradiction since, sup $A=n$ $∎$

3. **Field of Rational Numbers**

(3.1) **Theorem:** Every ordered field contains a field similar a field of rational number.

**Proof:** Let $(F,+, .)$ be ordered field. $n.1=1+1+…+1$ ($n$-times), to prove if $n.1=0⟹n=0$, let $k.1=0$ ($k\in Z\_{+}$), since $k.1=1+1+…+1$($k$-times)

$⟹k>1⟹k-1>0⟹\left(k-1\right).1>0⟹n.1\in F ∀ n\in Z\_{+}$ and $n.1=0$ iff $n=0$, also $m.1=n.1$ iff $m=n$. Since $(F,+, .)$ is a field $⟹-\left(n.1\right)\in F⟹-n.1=\left(-1\right)+\left(-1\right)+\cdots +(-1)$ ($n$-times)$ ⟹Z⊂F$, since $(F,+, .)$ is a field $⟹∀n\in Z$, $n\ne 0⟹\frac{1}{n}\in F⟹Q⊂F$ . $∎$

(3.2) **Theorem:** the equation $x^{2}=2$ has no root in $Q$.

**Proof:** Let$ y\in Q\ni y^{2}=2$, since $y\in Q⟹y=\frac{a}{b} \ni a,b\in Z$, $b\ne 0$ and $g.c.d\left(a,b\right)=1$. $y^{2}=2⟹\frac{a^{2}}{b^{2}}=2⟹a^{2}=2b^{2}…(1)$ $2b^{2}$is even number $⟹a^{2}$ is even number $⟹a$ is even number$⟹a=2c$ $⟹a^{2}=4c^{2}$, by (1)$ ⟹2b^{2}=4c^{2}⟹b^{2}$ is even number$⟹b$ is even number $⟹g.c.d\left(a,b\right)=2$, but this is contradiction$ ⟹y\notin Q$ $∎$

(3.3) **Theorem:** the equation $x^{2}=2$ has an unique positive real root.

(3.4) **Corollary:** The field of rational numbers is a proper subset of a field of real numbers $(Q⊂ R)$.

**Proof:** Since $x^{2}=2$ has a root $\sqrt{2 } ⟹\sqrt{2 } \in R$, $x^{2}=2$ has no root in $Q ⟹\sqrt{2 } \notin Q$. $∎$

(3.5) **Theorem:** The field of rational numbers is an incomplete.

**Proof:** Let $A=\{x\in Q: x^{2}<2\}⟹A\ne ∅$, let $y\in Q$ with sup $A=y⟹y^{2}=2$ or $y^{2}<2$ or $y^{2}>2$.

1. $y^{2}\ne 2$,
2. If $y^{2}<2$, put $z=\frac{4+3y}{3+2y}⟹z\in Q$, $z^{2}-2=\left(\frac{4+3y}{3+2y}\right)^{2}-2=\frac{y^{2}-2}{(3+2y)^{2}}<0$, ($y^{2}<2$)$ ⟹z^{2}<2⟹z\in A⟹z-y=\frac{4+3y}{3+2y}-y=\frac{2(2-y^{2})}{3+2y}>0⟹z>y$, this is contradiction, since $y$ is an upper bound of $A$.
3. If $y^{2}>2⟹z^{2}>2⟹z$ is an upper bound of $A$, this is contradiction, since $y$ is a smallest upper bound of $A$. $∎$

(3.6) **Theorem**(**Density of Rational Numbers**)

If $a,b\in R\ni a<b ∃ r\in Q\ni a<r<b$.

**Proof:** (1) $b-a>1$, put $A=\{n\in N:n>a\}$, since $a\in R⟹$ by Archimedes theorem $⟹∃m\in N\ni m>a⟹m\in A⟹A\ne ∅$. Since $N$ is a well ordered and $∅\ne A⊂N⟹A$ contains a smallest number $k$, since $k\in A⟹k>a$, since$ k$ is a smallest number in $A⟹k-1\notin A⟹k-1\leq a⟹k\leq a+1$, since $b-a>1⟹b>a+1⟹k<b⟹a<k<b⟹k\in Q$.

(2) If $a<0<b⟹0\in Q$.

(3) If $a<b<0⟹0<-b<-a$, by (1) $∃ r\in Q\ni -b<r<-a⟹a<-r<b⟹-r\in Q$. $∎$