**Real and Rational Numbers**

1. **Order Sets**
   1. **Definition**: Let be a relation on a set , we say that

* is a reflexive on , if .
* is a symmetric on , if , then .
* is a transitiveon , if , and , then .
* is an anti-symmetricon , if , and , then .
  1. **Note**: We say that is a preorderrelation on , if is a reflexive and transitive, and is a partial order relation on , if is a reflexive, transitive and anti-symmetric.
  2. **Note**: Let , then is a partial order set, we say that () is a partially order set.
  3. **Example**: The order pairs , , , , where are a partially order sets.
  4. **Definition**: Let be a partial order set, and we say are comparable, if or .
  5. **Definition**: Let , then is a totally ordered or chain in , if every two elements in are comparable.
  6. **Example**: The order pairs , are a totally ordered, but does not.
  7. **Definition**: Let be a partial order set and , we say that is a first element or a smallest element in , if . We say that is a last element or greatest element in , if .
  8. **Definition**: Let be a partial order set, then is called a well ordered, if every non empty subset of contains a first element.
  9. **Definition**: Let be a partial order set and , then is called a minimal element in , if , , then . We say that is a maximal element in , if , , then .
  10. **Examples**:

1. Let , then max and min.
2. Min and max does not exist.
3. Min and max does not exist.
4. Let , then max and min does not exist.
5. Let , then min and max does not exist.
6. Let , then max and min .
   1. **Definition**: Let be a partial order set and , we say that be lower bound of , if . We say that called a greatest lower bound of , if its:
7. A lower bound of ;
8. for all lower bound of .
   1. **Note**: We denote of element which a greatest lower bound of by inf .
   2. **Definition**: Let be a partial order set and , we say that be upper bound of , if . We say that called a smallest upper bound of , if its:
9. An upper bound of ;
10. for all upper bound of .
    1. **Note**: We denote of element which a smallest upper bound of by sup .
    2. **Examples**:
11. Let , then sup and inf does not exist.
12. Let , then sup and inf .
    1. **Definition**: Let be a partial order set and , we say that is a bounded below, if there exist a lower bound and is a bounded above, if there exists an upper bound. We say that bounded, if bounded from a lower and an upper.
    2. **Definition**: Let be a partial order set. We say that complete or complete ordered, if for all non empty subset and bounded from above in , then sup exists.
13. **Real Numbers**

(2.1) **Axioms of Field**

1. **Axioms of abelian.**

* .
* .

1. **Axioms of associative.**

* .

1. **Axiom of distribution.**
2. **Axioms of identity element.**

* There is such that .
* There is such that .

1. **Axioms of inverse element.**

* For all there is such that .
* For all , there is such that .

(2.2) **Theorem:** Let then

1. .
2. .
3. iff .
4. If , then iff .
5. iff or .
6. .
7. .
8. If , then and .

(2.3) **Axioms of order.**

There is a non-empty subset of which denoted by and its satisfy:

1. If then and .
2. If then one of following is true , , .

(2.4) **Definition**:

1. If then if .
2. means or .
3. means and .

(2.5) **Theorem:**

1. For all then either or or .
2. If and then .
3. iff .
4. If and then .
5. If then iff .
6. If then iff .
7. If and then .

(2.6) **The Completeness Axiom.**

Let then

1. If is an upper bounded, then sup exists.
2. If is a lower bounded, then inf exists.

(2.7) **Theorem:** Let and then

1. Inf iff
2. .
3. .
4. Sup iff
5. .
6. .

**Proof:** Let inf is a lower bound of satisfies.

Let , since is greatest lower bound of not lower bound of satisfies.

Now let (a), (b) are satisfy

(a) is a lower bound of , let . We must prove that not lower of . Put inf

(2) Assume that sup an upper bound of

Now to prove (2) let , since is a smallest upper bound of does not upper bound of .

(1) means is an upper bound of , let . Put by (2) sup

(2.8) **Theorem:**(Archimedes property)

If and then .

**Proof:** Let and . Put , , is an upper bound of bounded from above. Since satisfies the completeness sup . since is a smallest upper bound of does not upper bound of , since does not upper bound of contradiction

(2.9) **Corollary**:

1. .
2. .
3. .
4. a unique integer .

**Proof:** (1) Put and .

(2) Put and .

(3) since by (2) , now we must prove that . Put and is a lower bound (since satisfies completeness) inf , put .

(4) Put and has an upper bound, (since satisfies completeness) sup . To prove, suppose that , but this is contradiction since, sup

3. **Field of Rational Numbers**

(3.1) **Theorem:** Every ordered field contains a field similar a field of rational number.

**Proof:** Let be ordered field. (-times), to prove if , let (), since (-times)

and iff , also iff . Since is a field (-times), since is a field , .

(3.2) **Theorem:** the equation has no root in .

**Proof:** Let, since , and . is even number is even number is even number , by (1) is even number is even number , but this is contradiction

(3.3) **Theorem:** the equation has an unique positive real root.

(3.4) **Corollary:** The field of rational numbers is a proper subset of a field of real numbers .

**Proof:** Since has a root , has no root in .

(3.5) **Theorem:** The field of rational numbers is an incomplete.

**Proof:** Let , let with sup or or .

1. ,
2. If , put , , (), this is contradiction, since is an upper bound of .
3. If is an upper bound of , this is contradiction, since is a smallest upper bound of .

(3.6) **Theorem**(**Density of Rational Numbers**)

If .

**Proof:** (1) , put , since by Archimedes theorem . Since is a well ordered and contains a smallest number , since , since is a smallest number in , since .

(2) If .

(3) If , by (1) .