**2. Irrational and Real numbers**

Let $Q^{c}$ be a complement set of $Q$ in real number $R$. $Q^{c}=R\Q=\{x\in R:x\notin Q\}$, $Q^{c}$ is called the set of irrational numbers, since $Q^{c}\ne Q⟹\sqrt{2}\in Q^{c}$.

(2.1)**Theorem:** Let $x\in Q$ and $y\in Q^{c}$, then

1. $x+y\in Q^{c}$.
2. $xy\in Q^{c}$, with $x\ne 0$.

**Proof:**(1) Assume that $x+y\notin Q^{c}$, since $x+y\in R⟹x+y\in Q$, since $x\in Q$ and $Q$ is a field $⟹-x\in Q$, also $\left(x+y\right)+(-x)\in Q⟹y\in Q$, but this is a contradiction.

2)Let $xy\notin Q^{c}⟹xy\in Q$, since $Q$ is a field and $x\in Q$, $x\ne 0⟹\frac{1}{x}=x^{-1}\in Q$, also $\frac{1}{x}(xy)\in Q⟹y\in Q$, but this is a contradiction. $∎$

(2.2)**Theorem:**(Density of irrational numbers)

Let $a,b\in R\ni a<b ∃ s\in Q^{c}\ni a<s<b⟹∃ $an infinity irrational numbers between any two real numbers.

**Proof:** Since $a<b⟹a-\sqrt{2}<b-\sqrt{2}$, since $a-\sqrt{2}$ and $b-\sqrt{2}$ are real numbers $⟹$ by using density of rational numbers $⟹∃r\in Q\ni a-\sqrt{2}<r<b-\sqrt{2}⟹a<r+\sqrt{2}<b⟹$ since $r\in Q$ and $\sqrt{2}\in Q^{c}⟹s=r+\sqrt{2}\in Q^{c}⟹a<s<b$. Now, since $a<s⟹∃ s\_{1}\in Q\ni a<s\_{1}<s$, by continuing this operation, we get on an infinite number of irrational numbers located between$ a, b$. $∎$

(2.3) **Definition**: Let $a,b\in R\ni a<b$, then

$$\left(a,b\right)=\{x\in R:a<x<b\}$$

 $[a,b]=\{x\in R:a\leq x\leq b\}$

$$(a,b]=\{x\in R:a<x\leq b\}$$

$$[a,b)=\{x\in R:a\leq x<b\}$$

$$\left(-\infty ,b\right)=\{x\in R:-\infty <x<b\}$$

$$(-\infty ,b]=\{x\in R:-\infty <x\leq b\}$$

$$(a,\infty )=\{x\in R:a<x\leq \infty \}$$

$[a,\infty )=\{x\in R:a\leq x<\infty \}$.

(2.4) **Note:** According to density of rational and irrational numbers, we can say that every interval of real numbers contains an infinite number of rational and irrational.

(2.5) **Definition**: (**Absolute Value**) Let $x$ be a real number, absolute value of $x$ is denoted by $\left|x\right|$ and defined as:

$$\left|x\right|=\left\{\begin{array}{c} x, x\geq 0\\-x,x<0\end{array}\right.$$

(2.6)**Theorem:**(Properties of Absolute Value)

1. $\left|x\right|=$ max $\left\{-x,x\right\} ∀x\in R⟹\left|x\right|\geq -x, \left|x\right|\geq x$.
2. $\left|x\right|\geq 0 ∀ x\in R$.
3. $\left|x\right|=0$ iff $x=0$.
4. $\left|x\right|=\left|-x\right|∀ x\in R$.
5. $\left|x-y\right|=\left|y-x\right| ∀ x,y\in R$.
6. $\left|xy\right|=\left|x\right|\left|y\right| ∀ x,y\in R$.
7. $\left|\frac{x}{y}\right|=\frac{\left|x\right|}{\left|y\right|} ∀ x,y\in R, y\ne 0$.
8. $\left|x+y\right|\leq \left|x\right|+\left|y\right| ∀ x,y\in R$.
9. $\left|x-y\right|\leq \left|x\right|+\left|y\right| ∀ x,y\in R$.
10. $\left|\left|x\right|-\left|y\right|\right|\leq \left|x-y\right|∀ x,y\in R$.
11. $\left|x\right|\leq a$ iff $-a\leq x\leq a$.

**Some Important Inequalities**

(2.7)**Theorem**:

1. **Cauchy-schwars Inequality.**

If $p,q\in R\ni \frac{1}{p}+\frac{1}{q}=1⟹\sum\_{i=1}^{}\left|x\_{i}y\_{i}\right|\leq \left(\sum\_{i=1}^{}\left|x\_{i}\right|^{p}\right)^{\frac{1}{p}}\left(\sum\_{i=1}^{}\left|y\_{i}\right|^{p}\right)^{\frac{1}{p}}\ni x\_{i},y\_{i}\in R$. In particular, if $p=2⟹q=2$ and $\sum\_{i=1}^{}\left|x\_{i}y\_{i}\right|\leq \left(\sum\_{i=1}^{}\left|x\_{i}\right|^{2}\right)^{\frac{1}{2}}\left(\sum\_{i=1}^{}\left|y\_{i}\right|^{2}\right)^{\frac{1}{2}}$ .

1. **Minkokowsks Inequality.**

If $p\geq 1⟹(\sum\_{i=1}^{}\left|x\_{i}+y\_{i}\right|^{p})^{1/p}\leq \left(\sum\_{i=1}^{}\left|x\_{i}\right|^{p}\right)^{\frac{1}{p}}+\left(\sum\_{i=1}^{}\left|y\_{i}\right|^{p}\right)^{\frac{1}{p}}\ni x\_{i},y\_{i}\in R$.

**Countable Sets.**

(2.8) **Definition**: Let $A,B$ be a sets. We say that $A$ is an equivalent to $B$ and written by $A\~B$, if there is a bijective function from $A$ into $B$ and written$ A≁B$, if $A$ is an inequivalent to $B$.

(2.9)**Theorem**:

1. If $E=\{2,4,6,…\}$, then $N\~E$.

($f:N⟶E$ defined by $f\left(n\right)=2n ∀ n\in N$)

1. If $O=\{1,3,5,…\}$, then $N\~O$.

($f:N⟶O$ defined by $f\left(n\right)=2n-1 ∀ n\in N$)

1. If $N^{\*}=\{0,1,2,3,…\}$, then $N\~N^{\*}$.

($f:N⟶N^{\*}$ defined by $f\left(n\right)=n-1 ∀ n\in N$)

1. $Z\~N^{\*}$. ($f:Z⟶N^{\*}$ defined by $f(x)=\left\{\begin{array}{c} -2x , x\leq 0\\2x-1,x>0\end{array}\right.$)
2. $N\~Q$.

We deduce that$ E,O,N,N^{\*},Z,Q $ are equivalent.

1. If $A=[0,1]$, $I\_{1}=\left(a,b\right), I\_{2}=\left(a,b\right], I\_{3}=\left[a,b\right),I\_{4}=[a,b]$ then $A\~I\_{i}∀i=1,2,3,4$.

($f:A⟶I\_{i}$ defined by $f\left(x\right)=a+\left(b-a\right)x ∀ x\in A$)

1. If $A=(-1,1)$ and $B=(a,b)$, then $A\~B$.

($f:A⟶B$ defined by $f\left(x\right)=\frac{1}{2}\left(b-a\right)x+\frac{1}{2}\left(b+a\right) ∀ x\in A$)

1. If $A=(0,1)$ then $A\~R^{+}$.

($f:A⟶R^{+}$ defined by $f\left(x\right)=\frac{x}{1-x}∀ x\in A$)

1. If $A=(-1,1)$ then$ A\~R$.

($f:A⟶R$ defined by $f\left(x\right)=\sin(x)∀ x\in A$)

1. If $A=(\frac{-π}{2},\frac{π}{2})$ then $A\~R$.

($f:A⟶R$ defined by $f\left(x\right)=\tan(x)∀ x\in A$)

1. If $A=(0,1)$ then $A\~R$.
2. For all $k\in N$ put $N\_{k}=\left\{1,2,3,…,k\right\} $then
3. $N\_{k}≁N$.
4. $N\_{k}\~N\_{1}$ iff $k=1$.
5. $ P\left(X\right)≁X ∀$ set$ X$.

(2.10) **Definition**: Let $A$ be a set. We say that $A$ is a finite set, if $A$ is a non-empty set or equivalent to $N\_{k}$ for some $k\in N$. We say that $A$ is an infinite set, if $A$ does not finite set.

(2.11) **Definition**: If $A$ is a finite set, then $A\~N\_{k}$ for some $k\in N$ and then there is a bijective function $f:N\_{k}⟶A$, put $f\left(i\right)=a\_{i }∀i\in N\_{k}⟹a\_{i }\in A ∀i=1,2,3,…,k$ and then $A=\{a\_{1 }, a\_{2 }, …, a\_{k } \}$.

(2.12)**Theorem**:

1. Let $A, B$ be a non-empty sets such that $A\~B$ then
2. $A$ is a finite iff $B$ is a finite.
3. $A$ is an infinite iff $B$ is an infinite.
4. For all finite set inequivalent to proper subset.
5. Every subset of finite set be a finite.
6. If $A$ is an infinite set and $A⊂B$ then$ B$ is an infinite set.
7. If $A$ is an infinite set and $B$ is a set then $A∪B$ is an infinite.

(2.13) **Definition**: Let $A$ is a set. We say that $A$ be a countable set, if $A$ be a finite or equivalent to $N$. We say that $A$ be an infinite and countable, if $A$ be an infinite and equivalent to $N$. We say that $A$ be an uncountable, if $A$ be an infinite and inequivalent to $N$.

(2.14)**Theorem**:

1. Every finite set is a countable.
2. Each of $O,E,N, N^{\*},Z,Q$ be an infinite and countable set.
3. Each of $R$ and an intervals of $R$ are an uncountable sets.

(2.15) **Note**: If $A$ be an infinite countable set, then $N\~A$ and then there is a bijective function $f:N⟶A$, put$ f\left(n\right)=a\_{n }∀n\in N$ and then $A=\left\{a\_{n }:n\in N\right\}=\{a\_{1 },a\_{2 },a\_{3 },…\}$.

(2.16)**Theorem**:

1. Every countable infinite set be an equivalent to a proper subset.
2. Every infinite set contains a countable infinite subset.
3. The set $N×N$ be a countable.
4. If $A,B$ are a countable sets, then
5. $A∪B$ be a countable.
6. $A×B$ be a countable.