**8. Metric Topologies**

(8.1) **Definition**: Let be a metric space, and let . The set is called an open ball in , such is a center of a ball and is radius of a ball and denoted by .

(8.2) **Definition**: A closed ball with center and radius is denoted by .

(8.3) **Example**: In usual metric space, we have

1. Every open ball contains an open interval.
2. Every closed ball contains a closed interval.

**Solution:** (1) .

Let

.

(8.4) **Example**: Let and a function defined by . Discuss and .

**Solution:**

.

(8.5) **Example**: Discuss an open balls with the center and radius for following metric functions:

1. .
2. .
3. max .

**Solution:** (1)

.

(8.6) **Example**: Let be discrete metric space and let , then

1. If , then .
2. If , then .

**Solution:** (1) Let , since

, but .

(8.7) **Definition**: Let be a metric space and . We said that is an open set in , if .

(8.8) **Definition**: We say that is a closed set in , if is an open set in .

(8.9)**Theorem**: In any metric space, we have

1. Every open ball is an open set.
2. Every closed ball is a closed set.

**Proof:** (1) Let a metric space and .

We must prove that is an open set.

Let

Put , we must prove .

Let

Since

is an open set.

(8.10)**Corollary:** In usual metric space , we have

1. Every an open interval is an open set.
2. Every a closed interval is a closed set.

(8.11)**Theorem**: Let is a metric space and , then is an open iff equals to union of an open balls.

**Proof:** If , the proof will end.

If , let is an open set in .

equals to union of an open balls.

let equals to union of an open balls.

Since each open ball is an open set equals to union of an open set.

is an open set.

(8.12) **Example**: Prove that, every subset of discrete metric space is an open and closed.

**Solution:** Let is discrete metric space and . If , the proof will end.

If , let , take .

is an open set.

Let is an open set in is a closed set.

(8.13)**Theorem**: Let be a metric space.

1. Each of be an open sets in .
2. If be an open sets in , then be an open set in .
3. If is an open set in , then be an open set in .

**Proof:** (1) suppose that be a non- open set

, this is impossible, since does not contain on element is an open set.

Since be an open set.

(8.14) **Example**: Let be usual metric space, and let, we note that be an open set and be a non- open set.

(8.15)**Theorem**: Let be a metric space.

1. Each of be a closed sets in .
2. If be a closed sets in , then be a closed set in .
3. If is a closed set in , then be a closed set in .

**Proof:** (1) since and is an open set in is an open set in

is a closed set in

Since and is an open set in is an open set in

is a closed set in .

(8.16) **Example**: Let be usual metric space, and let, we note that be a closed set and be a non- closed set.

(8.17) **Notes**:

1. The point open ball with center contains on infinite number of points in .
2. .

(8.18)**Theorem**: In any metric space be every single set is a closed, every finite set be a closed.