**8. Metric Topologies**

 (8.1) **Definition**: Let $(X,d)$ be a metric space, $x\_{0}\in X$ and let $r\in R^{+}$. The set $\{x\in X:d(x,x\_{0})<r\}$ is called an open ball in $X$, such $x\_{0}$ is a center of a ball and $r$ is radius of a ball and denoted by $B\_{r}\left(x\_{0}\right)=\{x\in X:d(x,x\_{0})<r\}$.

(8.2) **Definition**: A closed ball with center $x\_{0}$ and radius $r$ is denoted by $\overbar{B\_{r}}\left(x\_{0}\right)=\{x\in X:d(x,x\_{0})\leq r\}$.

(8.3) **Example**: In usual metric space, we have

1. Every open ball contains an open interval.
2. Every closed ball contains a closed interval.

**Solution:** (1) $d\left(x,y\right)=\left|x-y\right| ∀x,y\in R$.

Let $x\_{0}\in R, r>0$

$$B\_{r}\left(x\_{0}\right)=\left\{x\in X:d\left(x,x\_{0}\right)<r\right\}=\{x\in X:\left|x-x\_{0}\right|<r\}$$

$=\left\{x\in X:-r<x-x\_{0}<r\right\}=\left\{x\in X:x\_{0}-r<x<x\_{0}+r\right\}=(x\_{0}-r,x\_{0}+r)$.

(8.4) **Example**: Let $X=[0,1]$ and a function $d:X×X\rightarrow R$ defined by $d\left(x,y\right)=\left|x-y\right| ∀x,y\in X$. Discuss $B\_{1}\left(\frac{1}{2}\right)$ and $B\_{\frac{1}{4}}\left(0\right)$.

**Solution:** $B\_{1}\left(\frac{1}{2}\right)=\left\{x\in X:d\left(x,\frac{1}{2}\right)<1\right\}=\left\{x\in X:\left|x-\frac{1}{2}\right|<1\right\}$

$=\left\{x\in X:\frac{-1}{2}<x<\frac{3}{2}\right\}=\left\{x\in X:0\leq x\leq 1\right\}=X$.

(8.5) **Example**: Discuss an open balls with the center $(0,0)$ and radius $1$ for following metric functions:

1. $d\_{1}\left(x,y\right)=\sqrt{(x\_{1}-y\_{1})^{2}+(x\_{2}-y\_{2})^{2}} ∀x=\left(x\_{1},x\_{2}\right), y=(y\_{1},y\_{2})\in R^{2}$.
2. $d\_{2}\left(x,y\right)=\left|x\_{1}-y\_{1}\right|+\left|x\_{2}-y\_{2}\right| ∀x=\left(x\_{1},x\_{2}\right), y=(y\_{1},y\_{2})\in R^{2}$.
3. $d\_{3}\left(x,y\right)= $ max $\{\left|x\_{1}-y\_{1}\right|,\left|x\_{2}-y\_{2}\right|\} ∀x=\left(x\_{1},x\_{2}\right), y=(y\_{1},y\_{2})\in R^{2}$.

**Solution:** (1) $r=1, (x\_{0},y\_{0})=(0,0)$

$B\_{r}\left(x\_{0}\right)=\left\{x\in X:d\left(x,x\_{0}\right)<r\right\}=\left\{\left(x\_{1},x\_{2}\right)\in R^{2}:x\_{1}^{2}+x\_{2}^{2}<1\right\}$.

(8.6) **Example**: Let $(X,d)$ be discrete metric space and let $x\_{0}\in X, r\in R^{+}$, then

1. If $r>1$, then $B\_{r}\left(x\_{0}\right)=X$.
2. If $r\leq 1$, then $B\_{r}\left(x\_{0}\right)=\{x\_{0}\}$.

**Solution:** (1) Let $x\in X$, since $d\left(x,x\_{0}\right)=\left\{\begin{array}{c}0, x=x\_{0}\\1, x\ne x\_{0}\end{array}\right.$

$⟹ d\left(x,x\_{0}\right)<r⟹x\in B\_{r}\left(x\_{0}\right)⟹X⊆B\_{r}\left(x\_{0}\right)$, but $B\_{r}\left(x\_{0}\right)⊆X⟹B\_{r}\left(x\_{0}\right)=X$.

(8.7) **Definition**: Let $(X,d)$ be a metric space and $A⊆X$. We said that $A$ is an open set in $X$, if $∀x\in X ∃r>0\ni B\_{r}\left(x\right)⊂A$.

(8.8) **Definition**: We say that $A$ is a closed set in $X$, if $A^{c}$ is an open set in $X$.

(8.9)**Theorem**: In any metric space, we have

1. Every open ball is an open set.
2. Every closed ball is a closed set.

**Proof:** (1) Let $(X,d)$ a metric space and $x\_{0}\in X, r>0$.

We must prove that $B\_{r}\left(x\_{0}\right)$ is an open set.

Let $x\in B\_{r}\left(x\_{0}\right)⟹d\left(x,x\_{0}\right)<r⟹r-d\left(x,x\_{0}\right)>0$

Put $r-d\left(x,x\_{0}\right)=r\_{1}⟹r\_{1}>0$, we must prove $B\_{r\_{1}}\left(x\_{0}\right)⊆B\_{r}\left(x\_{0}\right)$.

Let $y\in B\_{r\_{1}}\left(x\_{0}\right)⟹d\left(y,x\_{0}\right)<r\_{1}⟹d\left(y,x\_{0}\right)<r-d\left(y,x\_{0}\right)<r$

$$⟹d\left(y,x\right)+d\left(y,x\_{0}\right)<r$$

Since $d\left(y,x\_{0}\right)\leq d\left(y,x\right)+d\left(x,x\_{0}\right)⟹d\left(y,x\_{0}\right)<r⟹y\in B\_{r}\left(x\_{0}\right)$

$⟹B\_{r}\left(x\_{0}\right)$ is an open set.

(8.10)**Corollary:** In usual metric space $(R,d\_{u})$, we have

1. Every an open interval is an open set.
2. Every a closed interval is a closed set.

(8.11)**Theorem**: Let $(X,d)$ is a metric space and $A⊆X$, then $A$ is an open iff $A$ equals to union of an open balls.

**Proof:** If $A=∅$, the proof will end.

If $A\ne ∅$, let $A$ is an open set in $X$.

$$⟹∀x\in A ∃r\_{x}>0\ni B\_{r\_{x}}\left(x\right)⊆A⟹A⊆\bigcup\_{x\in A}^{}B\_{r\_{x}}\left(x\right)⊂A$$

$⟹A=\bigcup\_{x\in A}^{}B\_{r\_{x}}\left(x\right)⟹A$ equals to union of an open balls.

$⟸)$ let $A$ equals to union of an open balls.

Since each open ball is an open set $⟹A$ equals to union of an open set.

$⟹A$ is an open set.

(8.12) **Example**: Prove that, every subset of discrete metric space is an open and closed.

**Solution:** Let $(X,d)$ is discrete metric space and $A⊆X$. If $A=∅$, the proof will end.

If $A\ne ∅$, let $x\in A$, take $r=\frac{1}{2}$.

$$B\_{r}\left(x\right)=\left\{y\in X:d\left(y,x\right)<\frac{1}{2}\right\}=\left\{y\in X:d\left(y,x\right)=0\right\}=\left\{y\in X:y=x\right\}=\{x\}⊂A$$

$⟹A$ is an open set.

Let $B⊆X⟹B^{c}⊆X⟹B^{c}$ is an open set in $X⟹B$ is a closed set.

(8.13)**Theorem**: Let $(X,d)$ be a metric space.

1. Each of $∅,X$ be an open sets in $X$.
2. If $A\_{1},A\_{2} ,… , A\_{n}$ be an open sets in $X$, then $\bigcap\_{i=1}^{n}A\_{i}$ be an open set in $X$.
3. If $A\_{λ}∀ λ\in Λ$ is an open set in $X$, then $\bigcup\_{λ\in Λ}^{}A\_{λ}$ be an open set in $X$.

**Proof:** (1) suppose that $∅$ be a non- open set

 $⟹∃x\in ∅\ni B\_{r}\left(x\right)⊆∅ ∀r>0$, this is impossible, since $∅$ does not contain on element $⟹∅$ is an open set.

Since $B\_{r}\left(x\right)⊆X ∀x\in X, r>0⟹X$ be an open set.

(8.14) **Example**: Let $(R,d\_{u})$ be usual metric space, and let$ A\_{n}=\left(\frac{-1}{n},\frac{1}{n}\right)∀n\in Z^{+}$, we note that $A\_{n}$ be an open set $∀n\in Z^{+}$ and $\bigcap\_{i=1}^{\infty }A\_{n}=\{0\}$ be a non- open set.

(8.15)**Theorem**: Let $(X,d)$ be a metric space.

1. Each of $∅,X$ be a closed sets in $X$.
2. If $A\_{1},A\_{2} ,… , A\_{n}$ be a closed sets in $X$, then $\bigcup\_{i=1}^{n}A\_{i}$ be a closed set in $X$.
3. If $A\_{λ}∀ λ\in Λ$ is a closed set in $X$, then $\bigcap\_{λ\in Λ}^{}A\_{λ}$ be a closed set in $X$.

**Proof:** (1) since $∅^{c}=X$ and $X$ is an open set in $X⟹∅^{c}$ is an open set in $X$

$⟹∅$ is a closed set in $X$

Since $X^{c}=∅$ and $∅$ is an open set in $X⟹X^{c}$ is an open set in $X$

$⟹X$ is a closed set in $X$.

(8.16) **Example**: Let $(R,d\_{u})$ be usual metric space, and let$ A\_{n}=\left[\frac{1}{n},1\right]∀n\in Z^{+}$, we note that $A\_{n}$ be a closed set $∀n\in Z^{+}$ and $\bigcup\_{i=1}^{\infty }A\_{n}=(0,1]$ be a non- closed set.

(8.17) **Notes**:

1. The point $x\_{0}\in A^{'}⟺∀$ open ball with center $x\_{0}$ contains on infinite number of points in $A$.
2. $\overbar{A}=\{x\in X:d\left(x,A\right)=0\}$.

(8.18)**Theorem**: In any metric space $(X,d)$ be every single set is a closed, $⟹$ every finite set be a closed.