**10. Convergence**

 (10.1) **Definition**: Let $\{x\_{n}\}$ be a sequence in a metric space $(X,d)$. We said that $\{x\_{n}\}$ is a convergent in $X$, if $∃x\in X\ni ∀ε>0 ∃ k\in Z^{+}\ni d\left(x\_{n},x\right)<ε ∀n>k$.

$\lim\_{x\to \infty }x\_{n}=x$ or $x\_{n}\rightarrow x$ where $n\rightarrow \infty $

This means $d(x\_{n},x)\rightarrow 0⟺x\_{n}\rightarrow x$ where $n\rightarrow \infty $.

(10.2)**Theorem:** If $\{x\_{n}\}$ is a convergent in $(X,d)$, then the convergence point is a unique.

**Proof:** let $x\_{n}\rightarrow x$ and $x\_{n}\rightarrow y$ such that $x\ne y$ .

Let $d\left(x,y\right)=ε⟹ε>0$

Since $x\_{n}\rightarrow x⟹∃k\_{1}\in Z^{+}\ni d\left(x\_{n},x\right)<\frac{ε}{2} ∀n>k\_{1}$

Since $x\_{n}\rightarrow y⟹∃k\_{2}\in Z^{+}\ni d\left(x\_{n},y\right)<\frac{ε}{2} ∀n>k\_{2}$

Put $k=$ max $\left\{k\_{1},k\_{2}\right\}⟹d\left(x\_{n},x\right)<\frac{ε}{2}, d\left(x\_{n},y\right)<\frac{ε}{2}∀n>k$.

$ε=d\left(x,y\right)=d\left(x\_{n},x\right)+d\left(x\_{n},y\right)<\frac{ε}{2}+\frac{ε}{2}=ε$, but this is a contradiction $⟹x=y$.

(10.3)**Example:** Let $\{x\_{n}\}$ be a sequence in $(R,d\_{u})$, since $\{x\_{n}\}$ is a convergent

 $⟹∃x\in R\ni x\_{n}\rightarrow x$

Let $ε>0⟹∃k\in Z^{+}\ni d\left(x\_{n},x\right)<ε ∀n>k$

Since $(R,d\_{u})$ is an usual metric space $⟹d\left(x\_{n},x\right)=\left|x\_{n}-x\right|$

$$\left|x\_{n}-x\right|<ε ∀n>k⟹-ε<x\_{n}-x<ε ∀n>k⟹x-ε<x\_{n}<x+ε$$

$$⟹x\_{n}\in \left(x-ε,x+ε\right) ∀n>k$$

This means $\{x\_{n}\}$ is convergent in $R$, if $∃x\in R\ni ∀ε>0 ∃\left(x-ε,x+ε\right)$ with the center$ x$.

(10.4)**Theorem:** Let $A$ is a subset in $(X,d)$, then $x\in \overbar{A}$ $⟺$ $∃ \{x\_{n}\}$ in $A\ni x\_{n}\rightarrow x$.

(10.5) **Definition**: Let $\{x\_{n}\}$ be a sequence in a metric space $(X,d)$. We said that $\{x\_{n}\}$ is Cauchy sequence in $X$, if $∀ε>0 ∃ k\in Z^{+}\ni d\left(x\_{n},x\_{m}\right)<ε ∀n,m>k$.

(10.6)**Theorem:** Every convergent sequence$\{x\_{n}\}$ in a metric space $(X,d)$ be Cauchy sequence.

**Proof:** let $\{x\_{n}\}$ be a convergent sequence in $(X,d)⟹∃x\in X\ni x\_{n}\rightarrow x$

Let $ε>0$, since $x\_{n}\rightarrow x⟹∃ k\in Z^{+}\ni d\left(x\_{n},x\right)<\frac{ε}{2} ∀n>k$

If $n,m>k⟹d\left(x\_{n},x\right)<\frac{ε}{2}, d\left(x\_{m},x\right)<\frac{ε}{2}$

$$d\left(x\_{n},x\_{m}\right)\leq d\left(x\_{n},x\right)+d\left(x\_{m},x\right)<\frac{ε}{2}+\frac{ε}{2}=ε$$

$⟹\{x\_{n}\}$ is Cauchy sequence.

(10.7)**Note:** Not necessary that every Cauchy sequence in a metric space $(X,d)$ is a convergent, for example.

(10.8)**Example:** Let $X=R\\{0\}$, a function $d:X×X\rightarrow R$ defined by $d\left(x,y\right)=\left|x-y\right|$.

**Solution:** Let $x\_{n}=\frac{1}{n}$, we note that $(X,d)$ is a metric space and $\{x\_{n}\}$ be Cauchy sequence in $X$, but $\left\{x\_{n}\right\} $does not convergent to $x$.

(10.9)**Theorem:** Let $(X,d)$ a metric space and $\left\{x\_{n}\right\},\{y\_{n}\}$ in $X\ni x\_{n}\rightarrow x,y\_{n}\rightarrow y \ni x,y\in X$, then $d(x\_{n}, y\_{n})\rightarrow (x,y)$.

**Proof:** $d(x\_{n}, y)-d\left(x,y\right)=(d(x\_{n}, y\_{n})-d\left(x\_{n},y\right))+(d(x\_{n}, y)-d\left(x,y\right))$

$\left|d(x\_{n}, y)-d\left(x,y\right)\right|\leq \left|d(x\_{n}, y\_{n})-d\left(x\_{n},y\right)\right|+\left|d(x\_{n}, y)-d\left(x,y\right)\right|\leq d(x\_{n}, x)+d(y\_{n}, y)\rightarrow 0,n\rightarrow \infty ⟹d(x\_{n}, y\_{n})\rightarrow (x,y)$.

(10.10)**Example:** Let $(X,d)$ is a discrete metric space and $\{x\_{n}\}$ in $X$. Prove that $x\_{n}\rightarrow x\ni x\in X⟺∃k\in Z^{+}\ni x\_{n}=x ∀n>k$.

**Proof:** let $x\_{n}\rightarrow x⟹∀ε>0 ∃ k\in Z^{+}\ni d\left(x\_{n},x\right)<ε ∀n>k$.

Since $d\left(x,y\right)=\left\{\begin{array}{c}1, x\ne y\\0, x=y\end{array}\right.⟹d\left(x\_{n},x\right)=0 ∀n>k⟹x\_{n}=x∀n>k$.

(10.11)**Definition:** We said that $(X,d)$ is complete, if for all Cauchy sequence is a convergent.

(10.12)**Example:** Euclidean space $(R^{n},d)$ be complete metric space.

**Solution:** let $x,y\in R^{n},\ni x=\left(x\_{1},…, x\_{n}\right), y=(y\_{1},…, y\_{n}) $

$$d\left(x,y\right)=\sqrt{\sum\_{i=1}^{n}(x\_{i}-y\_{i})^{2}}$$

Let $\{x\_{m}\}$ be Cauchy sequence in $R^{n}$.

$$x\_{m}=\left(x\_{1}^{\left(m\right)},…,x\_{n}^{\left(m\right)}\right), x\_{m}\in R^{n} $$

Let $ε>0 ⟹∃ k\in Z^{+}\ni d\left(x\_{m},x\_{l}\right)=\sqrt{\sum\_{i=1}^{n}(x\_{i}^{(m)}-y\_{i}^{(l)})^{2}}$

$⟹\{x\_{m}\}$ Cauchy sequence in $R ∀i=1,…,n$

Since $R$ is complete field $⟹x\_{i}\in R\ni i=1,…,n ∀ x\_{i}^{\left(m\right)}\rightarrow x\_{i}$

Put $x=\left(x\_{1},…, x\_{n}\right)⟹x\in R^{n}⟹x\_{m}\rightarrow x$

So, $\{x\_{m}\}$ be convergent in $R^{n}$.