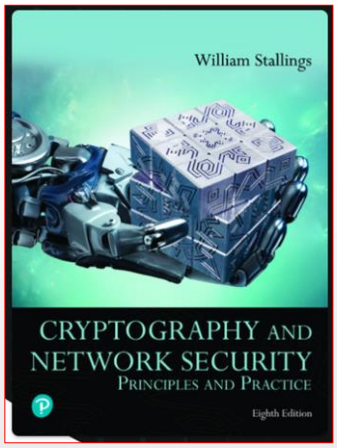
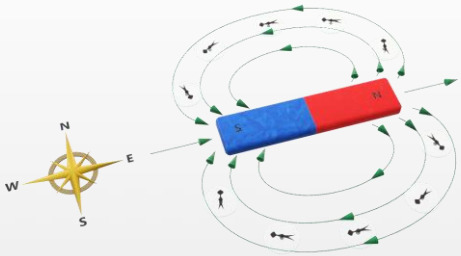
Bashar Al-Esawi

Information Security

M.Sc. Course

Prof. DR. Bashar M. Nema





Lesson overview

Information security (commonly known as InfoSec) refers to the procedures and practices that corporations use to protect their data.
Information security protects sensitive data from unauthorized acts such as scrutiny, modification, recording, disruption, or destruction. The goal is to secure and preserve the privacy of important data like client account information, financial information, or intellectual property.

Cyber security is the activity of securing computer systems, networks, devices, and applications from cyber attacks of any kind. Cyber security threats have risen above critical levels because of the inevitable spread of digital transformation, putting your sensitive data in jeopardy.
Because of its complexity in geopolitics and the more dispersed attack methods, corporations and national governments have begun to perceive cyber security as a key concern. Many firms increasingly include information risk management into their overall risk management strategy.

Learning objectives

- Know how to PROTECT (Data and Information).
- Know how to be a STANDARD APPS in Information security.
- Know how to PROTECT Recourses that connected online via concept of CYBERSECURITY.
- Know other ideas and concepts of these two fields.

EXPLAIN

Course Materials

Information Security		Cybersecurity	
Lectures	Seminars	Lectures	Seminars

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Sanjay Rao and Ruben Torres of Purdue University developed the laboratory exercises that appear in the IRC. The following people contributed project assignments that appear in the instructor's supplement: Henning Schulzrinne (Columbia University); Cetin Kaya Koc (Oregon State University); and David Balenson (Trusted Information Systems and George Washington University). Kim McLaughlin developed the test bank.

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ABOUT THE AUTHOR

Dr. William Stallings has authored 18 titles, and counting revised editions, over 40 books on computer security, computer networking, and computer architecture. His writings have appeared in numerous publications, including the *Proceedings of the IEEE, ACM Computing Reviews*, and *Cryptologia*.

He has 13 times received the award for the best Computer Science textbook of the year from the Text and Academic Authors Association.

In over 30 years in the field, he has been a technical contributor, technical manager, and an executive with several high-technology firms. He has designed and implemented both TCP/IP-based and OSI-based protocol suites on a variety of computers and operating systems, ranging from microcomputers to mainframes. As a consultant, he has advised government agencies, computer and software vendors, and major users on the design, selection, and use of networking software and products.

He created and maintains the *Computer Science Student Resource Site* at ComputerScienceStudent.com. This site provides documents and links on a variety of subjects of general interest to computer science students (and professionals). He is a member of the editorial board of *Cryptologia*, a scholarly journal devoted to all aspects of cryptology.

Dr. Stallings holds a PhD from MIT in computer science and a BS from Notre Dame in electrical engineering.

PART ONE: BACKGROUND

CHAPTER 1

COMPUTER AND NETWORK SECURITY CONCEPTS

Bashar Al-Esawi

LEARNING OBJECTIVES

After studying this chapter, you should be able to:

- Describe the key security requirements of confidentiality, integrity, and availability.
- Describe the X.800 security architecture for OSI.
- Discuss the types of security threats and attacks that must be dealt with and give examples of the types of threats and attacks that apply to different categories of computer and network assets.
- Explain the fundamental security design principles.
- Discuss the use of attack surfaces and attack trees.
- List and briefly describe key organizations involved in cryptography standards.

This book focuses on two broad areas: cryptographic algorithms and protocols, which have a broad range of applications; and network and Internet security, which rely heavily on cryptographic techniques.

Cryptographic algorithms and protocols can be grouped into four main areas:

- **Symmetric encryption:** Used to conceal the contents of blocks or streams of data of any size, including messages, files, encryption keys, and passwords.
- **Asymmetric encryption:** Used to conceal small blocks of data, such as encryption keys and hash function values, which are used in digital signatures.
- **Data integrity algorithms:** Used to protect blocks of data, such as messages, from alteration.
- **Authentication protocols:** These are schemes based on the use of cryptographic algorithms designed to authenticate the identity of entities.

The field of **network and Internet security** consists of measures to deter, prevent, detect, and correct security violations that involve the transmission of information. That is a broad statement that covers a host of possibilities. To give you a feel for the areas covered in this book, consider the following examples of security violations:

1. User A transmits a file to user B. The file contains sensitive information (e.g., payroll records) that is to be protected from disclosure. User C, who is not authorized to read the file, is able to monitor the transmission and capture a copy of the file during its transmission.
2. A network manager, D, transmits a message to a computer, E, under its management. The message instructs computer E to update an authorization file to include the identities of a number of new users who are to be given access to that computer. User F intercepts the message, alters its contents to add or delete entries, and then forwards the message to computer E, which accepts the message as coming from manager D and updates its authorization file accordingly.

3. Rather than intercept a message, user F constructs its own message with the desired entries and transmits that message to computer E as if it had come from manager D. Computer E accepts the message as coming from manager D and updates its authorization file accordingly.
4. An employee is fired without warning. The personnel manager sends a message to a server system to invalidate the employee's account. When the invalidation is accomplished, the server is to post a notice to the employee's file as confirmation of the action. The employee is able to intercept the message and delay it long enough to make a final access to the server to retrieve sensitive information. The message is then forwarded, the action taken, and the confirmation posted. The employee's action may go unnoticed for some considerable time.
5. A message is sent from a customer to a stockbroker with instructions for various transactions. Subsequently, the investments lose value and the customer denies sending the message.

Although this list by no means exhausts the possible types of network security violations, it illustrates the range of concerns of network security.

1.1 COMPUTER SECURITY CONCEPTS

A Definition of Computer Security

The NIST *Computer Security Handbook* [NIST95] defines the term *computer security* as follows:

> **Computer Security:** The protection afforded to an automated information system in order to attain the applicable objectives of preserving the integrity, availability, and confidentiality of information system resources (includes hardware, software, firmware, information/data, and telecommunications).

This definition introduces three key objectives that are at the heart of computer security:

- **Confidentiality:** This term covers two related concepts:

  Data[1] **confidentiality:** Assures that private or confidential information is not made available or disclosed to unauthorized individuals.

  Privacy: Assures that individuals control or influence what information related to them may be collected and stored and by whom and to whom that information may be disclosed.

[1]RFC 4949 defines *information* as "facts and ideas, which can be represented (encoded) as various forms of data," and *data* as "information in a specific physical representation, usually a sequence of symbols that have meaning; especially a representation of information that can be processed or produced by a computer." Security literature typically does not make much of a distinction, nor does this book.

- **Integrity:** This term covers two related concepts:

  Data integrity: Assures that information (both stored and in transmitted packets) and programs are changed only in a specified and authorized manner.

  System integrity: Assures that a system performs its intended function in an unimpaired manner, free from deliberate or inadvertent unauthorized manipulation of the system.

- **Availability:** Assures that systems work promptly and service is not denied to authorized users.

These three concepts form what is often referred to as the **CIA triad**. The three concepts embody the fundamental security objectives for both data and for information and computing services. For example, the NIST standard FIPS 199 (*Standards for Security Categorization of Federal Information and Information Systems*) lists confidentiality, integrity, and availability as the three security objectives for information and for information systems. FIPS 199 provides a useful characterization of these three objectives in terms of requirements and the definition of a loss of security in each category:

- **Confidentiality:** Preserving authorized restrictions on information access and disclosure, including means for protecting personal privacy and proprietary information. A loss of confidentiality is the unauthorized disclosure of information.
- **Integrity:** Guarding against improper information modification or destruction, including ensuring information nonrepudiation and authenticity. A loss of integrity is the unauthorized modification or destruction of information.
- **Availability:** Ensuring timely and reliable access to and use of information. A loss of availability is the disruption of access to or use of information or an information system.

Although the use of the CIA triad to define security objectives is well established, some in the security field feel that additional concepts are needed to present a complete picture (Figure 1.1). Two of the most commonly mentioned are as follows:

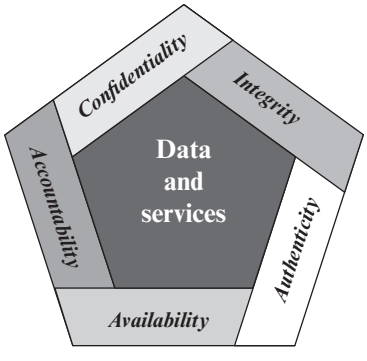

Figure 1.1 Essential Network and Computer Security Requirements

- **Authenticity:** The property of being genuine and being able to be verified and trusted; confidence in the validity of a transmission, a message, or message originator. This means verifying that users are who they say they are and that each input arriving at the system came from a trusted source.
- **Accountability:** The security goal that generates the requirement for actions of an entity to be traced uniquely to that entity. This supports nonrepudiation, deterrence, fault isolation, intrusion detection and prevention, and after-action recovery and legal action. Because truly secure systems are not yet an achievable goal, we must be able to trace a security breach to a responsible party. Systems must keep records of their activities to permit later forensic analysis to trace security breaches or to aid in transaction disputes.

Examples

We now provide some examples of applications that illustrate the requirements just enumerated.[2] For these examples, we use three levels of impact on organizations or individuals should there be a breach of security (i.e., a loss of confidentiality, integrity, or availability). These levels are defined in FIPS PUB 199:

- **Low:** The loss could be expected to have a limited adverse effect on organizational operations, organizational assets, or individuals. A limited adverse effect means that, for example, the loss of confidentiality, integrity, or availability might (i) cause a degradation in mission capability to an extent and duration that the organization is able to perform its primary functions, but the effectiveness of the functions is noticeably reduced; (ii) result in minor damage to organizational assets; (iii) result in minor financial loss; or (iv) result in minor harm to individuals.
- **Moderate:** The loss could be expected to have a serious adverse effect on organizational operations, organizational assets, or individuals. A serious adverse effect means that, for example, the loss might (i) cause a significant degradation in mission capability to an extent and duration that the organization is able to perform its primary functions, but the effectiveness of the functions is significantly reduced; (ii) result in significant damage to organizational assets; (iii) result in significant financial loss; or (iv) result in significant harm to individuals that does not involve loss of life or serious, life-threatening injuries.
- **High:** The loss could be expected to have a severe or catastrophic adverse effect on organizational operations, organizational assets, or individuals. A severe or catastrophic adverse effect means that, for example, the loss might (i) cause a severe degradation in or loss of mission capability to an extent and duration that the organization is not able to perform one or more of its primary functions; (ii) result in major damage to organizational assets; (iii) result in major financial loss; or (iv) result in severe or catastrophic harm to individuals involving loss of life or serious, life-threatening injuries.

[2]These examples are taken from a security policy document published by the Information Technology Security and Privacy Office at Purdue University.

Confidentiality Student grade information is an asset whose confidentiality is considered to be highly important by students. In the United States, the release of such information is regulated by the Family Educational Rights and Privacy Act (FERPA). Grade information should only be available to students, their parents, and employees that require the information to do their job. Student enrollment information may have a moderate confidentiality rating. While still covered by FERPA, this information is seen by more people on a daily basis, is less likely to be targeted than grade information, and results in less damage if disclosed. Directory information, such as lists of students or faculty or departmental lists, may be assigned a low confidentiality rating or indeed no rating. This information is typically freely available to the public and published on a school's Web site.

Integrity Several aspects of integrity are illustrated by the example of a hospital patient's allergy information stored in a database. The doctor should be able to trust that the information is correct and current. Now suppose that an employee (e.g., a nurse) who is authorized to view and update this information deliberately falsifies the data to cause harm to the hospital. The database needs to be restored to a trusted basis quickly, and it should be possible to trace the error back to the person responsible. Patient allergy information is an example of an asset with a high requirement for integrity. Inaccurate information could result in serious harm or death to a patient and expose the hospital to massive liability.

An example of an asset that may be assigned a moderate level of integrity requirement is a Web site that offers a forum to registered users to discuss some specific topic. Either a registered user or a hacker could falsify some entries or deface the Web site. If the forum exists only for the enjoyment of the users, brings in little or no advertising revenue, and is not used for something important such as research, then potential damage is not severe. The Web master may experience some data, financial, and time loss.

An example of a low integrity requirement is an anonymous online poll. Many Web sites, such as news organizations, offer these polls to their users with very few safeguards. However, the inaccuracy and unscientific nature of such polls is well understood.

Availability The more critical a component or service, the higher is the level of availability required. Consider a system that provides authentication services for critical systems, applications, and devices. An interruption of service results in the inability for customers to access computing resources and staff to access the resources they need to perform critical tasks. The loss of the service translates into a large financial loss in lost employee productivity and potential customer loss.

An example of an asset that would typically be rated as having a moderate availability requirement is a public Web site for a university; the Web site provides information for current and prospective students and donors. Such a site is not a critical component of the university's information system, but its unavailability will cause some embarrassment.

An online telephone directory lookup application would be classified as a low availability requirement. Although the temporary loss of the application may be an annoyance, there are other ways to access the information, such as a hardcopy directory or the operator.

The Challenges of Computer Security

Computer and network security is both fascinating and complex. Some of the reasons follow:

1. Security is not as simple as it might first appear to the novice. The requirements seem to be straightforward; indeed, most of the major requirements for security services can be given self-explanatory, one-word labels: confidentiality, authentication, nonrepudiation, or integrity. But the mechanisms used to meet those requirements can be quite complex, and understanding them may involve rather subtle reasoning.
2. In developing a particular security mechanism or algorithm, one must always consider potential attacks on those security features. In many cases, successful attacks are designed by looking at the problem in a completely different way, therefore exploiting an unexpected weakness in the mechanism.
3. Because of point 2, the procedures used to provide particular services are often counterintuitive. Typically, a security mechanism is complex, and it is not obvious from the statement of a particular requirement that such elaborate measures are needed. It is only when the various aspects of the threat are considered that elaborate security mechanisms make sense.
4. Having designed various security mechanisms, it is necessary to decide where to use them. This is true both in terms of physical placement (e.g., at what points in a network are certain security mechanisms needed) and in a logical sense (e.g., at what layer or layers of an architecture such as TCP/IP [Transmission Control Protocol/Internet Protocol] should mechanisms be placed).
5. Security mechanisms typically involve more than a particular algorithm or protocol. They also require that participants be in possession of some secret information (e.g., an encryption key), which raises questions about the creation, distribution, and protection of that secret information. There also may be a reliance on communications protocols whose behavior may complicate the task of developing the security mechanism. For example, if the proper functioning of the security mechanism requires setting time limits on the transit time of a message from sender to receiver, then any protocol or network that introduces variable, unpredictable delays may render such time limits meaningless.
6. Computer and network security is essentially a battle of wits between a perpetrator who tries to find holes and the designer or administrator who tries to close them. The great advantage that the attacker has is that he or she need only find a single weakness, while the designer must find and eliminate all weaknesses to achieve perfect security.
7. There is a natural tendency on the part of users and system managers to perceive little benefit from security investment until a security failure occurs.
8. Security requires regular, even constant, monitoring, and this is difficult in today's short-term, overloaded environment.
9. Security is still too often an afterthought to be incorporated into a system after the design is complete rather than being an integral part of the design process.

10. Many users and even security administrators view strong security as an impediment to efficient and user-friendly operation of an information system or use of information.

The difficulties just enumerated will be encountered in numerous ways as we examine the various security threats and mechanisms throughout this book.

1.2 THE OSI SECURITY ARCHITECTURE

To assess effectively the security needs of an organization and to evaluate and choose various security products and policies, the manager responsible for security needs some systematic way of defining the requirements for security and characterizing the approaches to satisfying those requirements. This is difficult enough in a centralized data processing environment; with the use of local and wide area networks, the problems are compounded.

ITU-T[3] Recommendation X.800, *Security Architecture for OSI*, defines such a systematic approach.[4] The OSI security architecture is useful to managers as a way of organizing the task of providing security. Furthermore, because this architecture was developed as an international standard, computer and communications vendors have developed security features for their products and services that relate to this structured definition of services and mechanisms.

For our purposes, the OSI security architecture provides a useful, if abstract, overview of many of the concepts that this book deals with. The OSI security architecture focuses on security attacks, mechanisms, and services. These can be defined briefly as

- **Security attack:** Any action that compromises the security of information owned by an organization.
- **Security mechanism:** A process (or a device incorporating such a process) that is designed to detect, prevent, or recover from a security attack.
- **Security service:** A processing or communication service that enhances the security of the data processing systems and the information transfers of an organization. The services are intended to counter security attacks, and they make use of one or more security mechanisms to provide the service.

In the literature, the terms *threat* and *attack* are commonly used to mean more or less the same thing. Table 1.1 provides definitions taken from RFC 4949, *Internet Security Glossary*.

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[3]The International Telecommunication Union (ITU) Telecommunication Standardization Sector (ITU-T) is a United Nations-sponsored agency that develops standards, called Recommendations, relating to telecommunications and to open systems interconnection (OSI).

[4]The OSI security architecture was developed in the context of the OSI protocol architecture, which is described in Appendix L. However, for our purposes in this chapter, an understanding of the OSI protocol architecture is not required.

Table 1.1 Threats and Attacks (RFC 4949)

Threat A potential for violation of security, which exists when there is a circumstance, capability, action, or event that could breach security and cause harm. That is, a threat is a possible danger that might exploit a vulnerability.
Attack An assault on system security that derives from an intelligent threat; that is, an intelligent act that is a deliberate attempt (especially in the sense of a method or technique) to evade security services and violate the security policy of a system.

1.3 SECURITY ATTACKS

A useful means of classifying security attacks, used both in X.800 and RFC 4949, is in terms of *passive attacks* and *active attacks* (Figure 1.2). A passive attack attempts to learn or make use of information from the system but does not affect system resources. An active attack attempts to alter system resources or affect their operation.

Passive Attacks

Passive attacks (Figure 1.2a) are in the nature of eavesdropping on, or monitoring of, transmissions. The goal of the opponent is to obtain information that is being transmitted. Two types of passive attacks are the release of message contents and traffic analysis.

The **release of message contents** is easily understood. A telephone conversation, an electronic mail message, and a transferred file may contain sensitive or confidential information. We would like to prevent an opponent from learning the contents of these transmissions.

A second type of passive attack, **traffic analysis**, is subtler. Suppose that we had a way of masking the contents of messages or other information traffic so that opponents, even if they captured the message, could not extract the information from the message. The common technique for masking contents is encryption. If we had encryption protection in place, an opponent might still be able to observe the pattern of these messages. The opponent could determine the location and identity of communicating hosts and could observe the frequency and length of messages being exchanged. This information might be useful in guessing the nature of the communication that was taking place.

Passive attacks are very difficult to detect, because they do not involve any alteration of the data. Typically, the message traffic is sent and received in an apparently normal fashion, and neither the sender nor receiver is aware that a third party has read the messages or observed the traffic pattern. However, it is feasible to prevent the success of these attacks, usually by means of encryption. Thus, the emphasis in dealing with passive attacks is on prevention rather than detection.

Active Attacks

Active attacks (Figure 1.2b) involve some modification of the data stream or the creation of a false stream and can be subdivided into four categories: masquerade, replay, modification of messages, and denial of service.

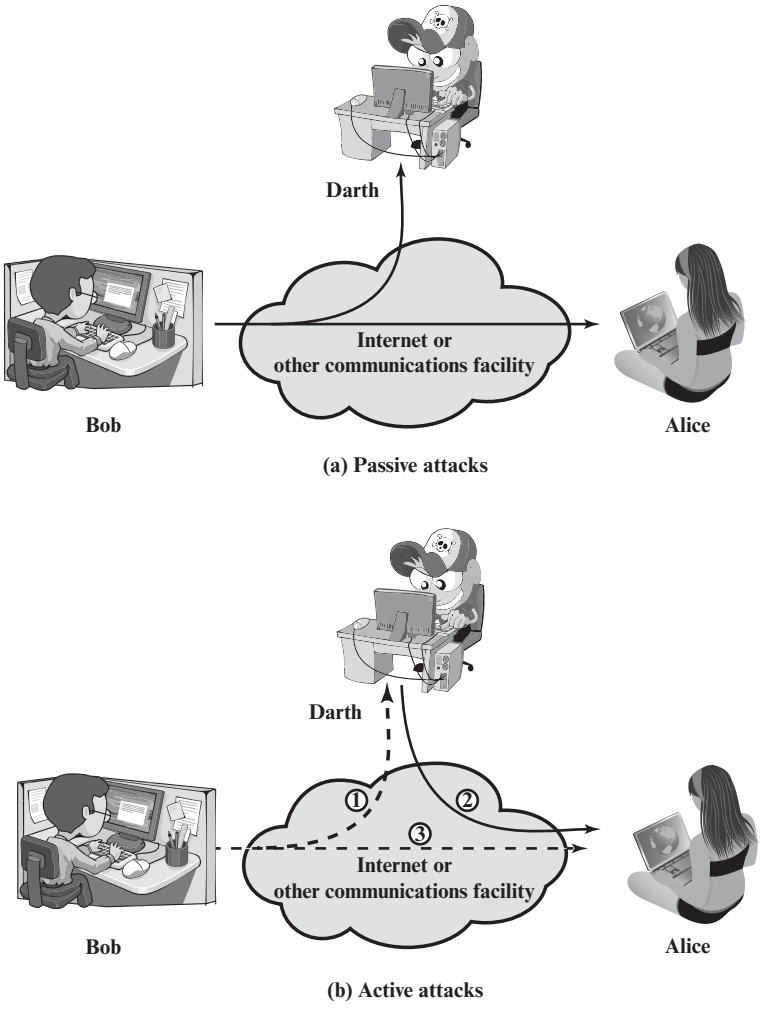

Figure 1.2 Security Attacks

A **masquerade** takes place when one entity pretends to be a different entity (path 2 of Figure 1.2b is active). A masquerade attack usually includes one of the other forms of active attack. For example, authentication sequences can be captured and replayed after a valid authentication sequence has taken place, thus enabling an authorized entity with few privileges to obtain extra privileges by impersonating an entity that has those privileges.

Replay involves the passive capture of a data unit and its subsequent retransmission to produce an unauthorized effect (paths 1, 2, and 3 active).

Modification of messages simply means that some portion of a legitimate message is altered, or that messages are delayed or reordered, to produce an unauthorized effect (paths 1 and 2 active). For example, a message meaning "Allow John Smith to read confidential file *accounts*" is modified to mean "Allow Fred Brown to read confidential file *accounts.*"

The **denial of service** prevents or inhibits the normal use or management of communications facilities (path 3 active). This attack may have a specific target; for example, an entity may suppress all messages directed to a particular destination (e.g., the security audit service). Another form of service denial is the disruption of an entire network, either by disabling the network or by overloading it with messages so as to degrade performance.

Active attacks present the opposite characteristics of passive attacks. Whereas passive attacks are difficult to detect, measures are available to prevent their success. On the other hand, it is quite difficult to prevent active attacks absolutely because of the wide variety of potential physical, software, and network vulnerabilities. Instead, the goal is to detect active attacks and to recover from any disruption or delays caused by them. If the detection has a deterrent effect, it may also contribute to prevention.

1.4 SECURITY SERVICES

X.800 defines a security service as a service that is provided by a protocol layer of communicating open systems and that ensures adequate security of the systems or of data transfers. Perhaps a clearer definition is found in RFC 4949, which provides the following definition: a processing or communication service that is provided by a system to give a specific kind of protection to system resources; security services implement security policies and are implemented by security mechanisms.

X.800 divides these services into five categories and fourteen specific services (Table 1.2). We look at each category in turn.[5]

Authentication

The authentication service is concerned with assuring that a communication is authentic. In the case of a single message, such as a warning or alarm signal, the function of the authentication service is to assure the recipient that the message is from the source that it claims to be from. In the case of an ongoing interaction, such as the connection of a terminal to a host, two aspects are involved. First, at the time of connection initiation, the service assures that the two entities are authentic, that is, that each is the entity that it claims to be. Second, the service must assure that the connection is not interfered with in such a way that a third party can masquerade as one of the two legitimate parties for the purposes of unauthorized transmission or reception.

Two specific authentication services are defined in X.800:

- **Peer entity authentication:** Provides for the corroboration of the identity of a peer entity in an association. Two entities are considered peers if they implement to same protocol in different systems; for example two TCP modules in two communicating systems. Peer entity authentication is provided for

[5]There is no universal agreement about many of the terms used in the security literature. For example, the term *integrity* is sometimes used to refer to all aspects of information security. The term *authentication* is sometimes used to refer both to verification of identity and to the various functions listed under integrity in this chapter. Our usage here agrees with both X.800 and RFC 4949.

Table 1.2 Security Services (X.800)

AUTHENTICATION The assurance that the communicating entity is the one that it claims to be. **Peer Entity Authentication** Used in association with a logical connection to provide confidence in the identity of the entities connected. **Data-Origin Authentication** In a connectionless transfer, provides assurance that the source of received data is as claimed. **ACCESS CONTROL** The prevention of unauthorized use of a resource (i.e., this service controls who can have access to a resource, under what conditions access can occur, and what those accessing the resource are allowed to do). **DATA CONFIDENTIALITY** The protection of data from unauthorized disclosure. **Connection Confidentiality** The protection of all user data on a connection. **Connectionless Confidentiality** The protection of all user data in a single data block. **Selective-Field Confidentiality** The confidentiality of selected fields within the user data on a connection or in a single data block. **Traffic-Flow Confidentiality** The protection of the information that might be derived from observation of traffic flows.	**DATA INTEGRITY** The assurance that data received are exactly as sent by an authorized entity (i.e., contain no modification, insertion, deletion, or replay). **Connection Integrity with Recovery** Provides for the integrity of all user data on a connection and detects any modification, insertion, deletion, or replay of any data within an entire data sequence, with recovery attempted. **Connection Integrity without Recovery** As above, but provides only detection without recovery. **Selective-Field Connection Integrity** Provides for the integrity of selected fields within the user data of a data block transferred over a connection and takes the form of determination of whether the selected fields have been modified, inserted, deleted, or replayed. **Connectionless Integrity** Provides for the integrity of a single connectionless data block and may take the form of detection of data modification. Additionally, a limited form of replay detection may be provided. **Selective-Field Connectionless Integrity** Provides for the integrity of selected fields within a single connectionless data block; takes the form of determination of whether the selected fields have been modified. **NONREPUDIATION** Provides protection against denial by one of the entities involved in a communication of having participated in all or part of the communication. **Nonrepudiation, Origin** Proof that the message was sent by the specified party. **Nonrepudiation, Destination** Proof that the message was received by the specified party.

use at the establishment of, or at times during the data transfer phase of, a connection. It attempts to provide confidence that an entity is not performing either a masquerade or an unauthorized replay of a previous connection.

- **Data origin authentication:** Provides for the corroboration of the source of a data unit. It does not provide protection against the duplication or modification of data units. This type of service supports applications like electronic mail, where there are no prior interactions between the communicating entities.

Access Control

In the context of network security, access control is the ability to limit and control the access to host systems and applications via communications links. To achieve this, each entity trying to gain access must first be identified, or authenticated, so that access rights can be tailored to the individual.

Data Confidentiality

Confidentiality is the protection of transmitted data from passive attacks. With respect to the content of a data transmission, several levels of protection can be identified. The broadest service protects all user data transmitted between two users over a period of time. For example, when a TCP connection is set up between two systems, this broad protection prevents the release of any user data transmitted over the TCP connection. Narrower forms of this service can also be defined, including the protection of a single message or even specific fields within a message. These refinements are less useful than the broad approach and may even be more complex and expensive to implement.

The other aspect of confidentiality is the protection of traffic flow from analysis. This requires that an attacker not be able to observe the source and destination, frequency, length, or other characteristics of the traffic on a communications facility.

Data Integrity

As with confidentiality, integrity can apply to a stream of messages, a single message, or selected fields within a message. Again, the most useful and straightforward approach is total stream protection.

A connection-oriented integrity service, one that deals with a stream of messages, assures that messages are received as sent with no duplication, insertion, modification, reordering, or replays. The destruction of data is also covered under this service. Thus, the connection-oriented integrity service addresses both message stream modification and denial of service. On the other hand, a connectionless integrity service, one that deals with individual messages without regard to any larger context, generally provides protection against message modification only.

We can make a distinction between service with and without recovery. Because the integrity service relates to active attacks, we are concerned with detection rather than prevention. If a violation of integrity is detected, then the service may simply report this violation, and some other portion of software or human intervention is required to recover from the violation. Alternatively, there are mechanisms available to recover from the loss of integrity of data, as we will review subsequently. The incorporation of automated recovery mechanisms is, in general, the more attractive alternative.

Nonrepudiation

Nonrepudiation prevents either sender or receiver from denying a transmitted message. Thus, when a message is sent, the receiver can prove that the alleged sender in fact sent the message. Similarly, when a message is received, the sender can prove that the alleged receiver in fact received the message.

Availability Service

Both X.800 and RFC 4949 define availability to be the property of a system or a system resource being accessible and usable upon demand by an authorized system entity, according to performance specifications for the system (i.e., a system is available if it provides services according to the system design whenever users request them). A variety of attacks can result in the loss of or reduction in availability. Some of these attacks are amenable to automated countermeasures, such as authentication and encryption, whereas others require some sort of physical action to prevent or recover from loss of availability of elements of a distributed system.

X.800 treats availability as a property to be associated with various security services. However, it makes sense to call out specifically an availability service. An availability service is one that protects a system to ensure its availability. This service addresses the security concerns raised by denial-of-service attacks. It depends on proper management and control of system resources and thus depends on access control service and other security services.

1.5 SECURITY MECHANISMS

Table 1.3 lists the security mechanisms defined in X.800. The mechanisms are divided into those that are implemented in a specific protocol layer, such as TCP or an application-layer protocol, and those that are not specific to any particular protocol layer or security service. These mechanisms will be covered in the appropriate

Table 1.3 Security Mechanisms (X.800)

SPECIFIC SECURITY MECHANISMS May be incorporated into the appropriate protocol layer in order to provide some of the OSI security services.	PERVASIVE SECURITY MECHANISMS Mechanisms that are not specific to any particular OSI security service or protocol layer.
Encipherment The use of mathematical algorithms to transform data into a form that is not readily intelligible. The transformation and subsequent recovery of the data depend on an algorithm and zero or more encryption keys.	**Trusted Functionality** That which is perceived to be correct with respect to some criteria (e.g., as established by a security policy).
Digital Signature Data appended to, or a cryptographic transformation of, a data unit that allows a recipient of the data unit to prove the source and integrity of the data unit and protect against forgery (e.g., by the recipient).	**Security Label** The marking bound to a resource (which may be a data unit) that names or designates the security attributes of that resource.
Access Control A variety of mechanisms that enforce access rights to resources.	**Event Detection** Detection of security-relevant events.
Data Integrity A variety of mechanisms used to assure the integrity of a data unit or stream of data units.	**Security Audit Trail** Data collected and potentially used to facilitate a security audit, which is an independent review and examination of system records and activities.
	Security Recovery Deals with requests from mechanisms, such as event handling and management functions, and takes recovery actions.

SPECIFIC SECURITY MECHANISMS

Authentication Exchange
A mechanism intended to ensure the identity of an entity by means of information exchange.

Traffic Padding
The insertion of bits into gaps in a data stream to frustrate traffic analysis attempts.

Routing Control
Enables selection of particular physically secure routes for certain data and allows routing changes, especially when a breach of security is suspected.

Notarization
The use of a trusted third party to assure certain properties of a data exchange.

places in the book. So we do not elaborate now, except to comment on the definition of encipherment. X.800 distinguishes between reversible encipherment mechanisms and irreversible encipherment mechanisms. A reversible encipherment mechanism is simply an encryption algorithm that allows data to be encrypted and subsequently decrypted. Irreversible encipherment mechanisms include hash algorithms and message authentication codes, which are used in digital signature and message authentication applications.

Table 1.4, based on one in X.800, indicates the relationship between security services and security mechanisms.

Table 1.4 Relationship Between Security Services and Mechanisms

SERVICE \ MECHANISM	Encipherment	Digital signature	Access control	Data integrity	Authentication exchange	Traffic padding	Routing control	Notarization
Peer entity authentication	Y	Y			Y			
Data origin authentication	Y	Y						
Access control			Y					
Confidentiality	Y						Y	
Traffic flow confidentiality	Y					Y	Y	
Data integrity	Y	Y		Y				
Nonrepudiation		Y		Y				Y
Availability				Y	Y			

1.6 FUNDAMENTAL SECURITY DESIGN PRINCIPLES

Despite years of research and development, it has not been possible to develop security design and implementation techniques that systematically exclude security flaws and prevent all unauthorized actions. In the absence of such foolproof techniques, it is useful to have a set of widely agreed design principles that can guide the development of protection mechanisms. The National Centers of Academic Excellence in Information Assurance/Cyber Defense, which is jointly sponsored by the U.S. National Security Agency and the U.S. Department of Homeland Security, list the following as fundamental security design principles [NCAE13]:

- Economy of mechanism
- Fail-safe defaults
- Complete mediation
- Open design
- Separation of privilege
- Least privilege
- Least common mechanism
- Psychological acceptability
- Isolation
- Encapsulation
- Modularity
- Layering
- Least astonishment

The first eight listed principles were first proposed in [SALT75] and have withstood the test of time. In this section, we briefly discuss each principle.

Economy of mechanism means that the design of security measures embodied in both hardware and software should be as simple and small as possible. The motivation for this principle is that relatively simple, small design is easier to test and verify thoroughly. With a complex design, there are many more opportunities for an adversary to discover subtle weaknesses to exploit that may be difficult to spot ahead of time. The more complex the mechanism, the more likely it is to possess exploitable flaws. Simple mechanisms tend to have fewer exploitable flaws and require less maintenance. Further, because configuration management issues are simplified, updating or replacing a simple mechanism becomes a less intensive process. In practice, this is perhaps the most difficult principle to honor. There is a constant demand for new features in both hardware and software, complicating the security design task. The best that can be done is to keep this principle in mind during system design to try to eliminate unnecessary complexity.

Fail-safe defaults means that access decisions should be based on permission rather than exclusion. That is, the default situation is lack of access, and the protection scheme identifies conditions under which access is permitted. This approach

exhibits a better failure mode than the alternative approach, where the default is to permit access. A design or implementation mistake in a mechanism that gives explicit permission tends to fail by refusing permission, a safe situation that can be quickly detected. On the other hand, a design or implementation mistake in a mechanism that explicitly excludes access tends to fail by allowing access, a failure that may long go unnoticed in normal use. Most file access systems and virtually all protected services on client/server systems use fail-safe defaults.

Complete mediation means that every access must be checked against the access control mechanism. Systems should not rely on access decisions retrieved from a cache. In a system designed to operate continuously, this principle requires that, if access decisions are remembered for future use, careful consideration be given to how changes in authority are propagated into such local memories. File access systems appear to provide an example of a system that complies with this principle. However, typically, once a user has opened a file, no check is made to see if permissions change. To fully implement complete mediation, every time a user reads a field or record in a file, or a data item in a database, the system must exercise access control. This resource-intensive approach is rarely used.

Open design means that the design of a security mechanism should be open rather than secret. For example, although encryption keys must be secret, encryption algorithms should be open to public scrutiny. The algorithms can then be reviewed by many experts, and users can therefore have high confidence in them. This is the philosophy behind the National Institute of Standards and Technology (NIST) program of standardizing encryption and hash algorithms, and has led to the widespread adoption of NIST-approved algorithms.

Separation of privilege is defined in [SALT75] as a practice in which multiple privilege attributes are required to achieve access to a restricted resource. A good example of this is multifactor user authentication, which requires the use of multiple techniques, such as a password and a smart card, to authorize a user. The term is also now applied to any technique in which a program is divided into parts that are limited to the specific privileges they require in order to perform a specific task. This is used to mitigate the potential damage of a computer security attack. One example of this latter interpretation of the principle is removing high privilege operations to another process and running that process with the higher privileges required to perform its tasks. Day-to-day interfaces are executed in a lower privileged process.

Least privilege means that every process and every user of the system should operate using the least set of privileges necessary to perform the task. A good example of the use of this principle is role-based access control. The system security policy can identify and define the various roles of users or processes. Each role is assigned only those permissions needed to perform its functions. Each permission specifies a permitted access to a particular resource (such as read and write access to a specified file or directory, connect access to a given host and port). Unless a permission is granted explicitly, the user or process should not be able to access the protected resource. More generally, any access control system should allow each user only the privileges that are authorized for that user. There is also a temporal aspect to the least privilege principle. For example, system programs or administrators who have special privileges should have those privileges only when necessary;

when they are doing ordinary activities the privileges should be withdrawn. Leaving them in place just opens the door to accidents.

Least common mechanism means that the design should minimize the functions shared by different users, providing mutual security. This principle helps reduce the number of unintended communication paths and reduces the amount of hardware and software on which all users depend, thus making it easier to verify if there are any undesirable security implications.

Psychological acceptability implies that the security mechanisms should not interfere unduly with the work of users, while at the same time meeting the needs of those who authorize access. If security mechanisms hinder the usability or accessibility of resources, then users may opt to turn off those mechanisms. Where possible, security mechanisms should be transparent to the users of the system or at most introduce minimal obstruction. In addition to not being intrusive or burdensome, security procedures must reflect the user's mental model of protection. If the protection procedures do not make sense to the user or if the user must translate his image of protection into a substantially different protocol, the user is likely to make errors.

Isolation is a principle that applies in three contexts. First, public access systems should be isolated from critical resources (data, processes, etc.) to prevent disclosure or tampering. In cases where the sensitivity or criticality of the information is high, organizations may want to limit the number of systems on which that data is stored and isolate them, either physically or logically. Physical isolation may include ensuring that no physical connection exists between an organization's public access information resources and an organization's critical information. When implementing logical isolation solutions, layers of security services and mechanisms should be established between public systems and secure systems responsible for protecting critical resources. Second, the processes and files of individual users should be isolated from one another except where it is explicitly desired. All modern operating systems provide facilities for such isolation, so that individual users have separate, isolated process space, memory space, and file space, with protections for preventing unauthorized access. And finally, security mechanisms should be isolated in the sense of preventing access to those mechanisms. For example, logical access control may provide a means of isolating cryptographic software from other parts of the host system and for protecting cryptographic software from tampering and the keys from replacement or disclosure.

Encapsulation can be viewed as a specific form of isolation based on object-oriented functionality. Protection is provided by encapsulating a collection of procedures and data objects in a domain of its own so that the internal structure of a data object is accessible only to the procedures of the protected subsystem, and the procedures may be called only at designated domain entry points.

Modularity in the context of security refers both to the development of security functions as separate, protected modules and to the use of a modular architecture for mechanism design and implementation. With respect to the use of separate security modules, the design goal here is to provide common security functions and services, such as cryptographic functions, as common modules. For example, numerous protocols and applications make use of cryptographic functions. Rather than implementing such functions in each protocol or application, a more secure design is provided by developing a common cryptographic module that can be invoked by numerous

protocols and applications. The design and implementation effort can then focus on the secure design and implementation of a single cryptographic module and including mechanisms to protect the module from tampering. With respect to the use of a modular architecture, each security mechanism should be able to support migration to new technology or upgrade of new features without requiring an entire system redesign. The security design should be modular so that individual parts of the security design can be upgraded without the requirement to modify the entire system.

Layering refers to the use of multiple, overlapping protection approaches addressing the people, technology, and operational aspects of information systems. By using multiple, overlapping protection approaches, the failure or circumvention of any individual protection approach will not leave the system unprotected. We will see throughout this book that a layering approach is often used to provide multiple barriers between an adversary and protected information or services. This technique is often referred to as *defense in depth*.

Least astonishment means that a program or user interface should always respond in the way that is least likely to astonish the user. For example, the mechanism for authorization should be transparent enough to a user that the user has a good intuitive understanding of how the security goals map to the provided security mechanism.

1.7 ATTACK SURFACES AND ATTACK TREES

In Section 1.3, we provided an overview of the spectrum of security threats and attacks facing computer and network systems. Section 22.1 goes into more detail about the nature of attacks and the types of adversaries that present security threats. In this section, we elaborate on two concepts that are useful in evaluating and classifying threats: attack surfaces and attack trees.

Attack Surfaces

An attack surface consists of the reachable and exploitable vulnerabilities in a system [MANA11, HOWA03]. Examples of attack surfaces are the following:

- Open ports on outward facing Web and other servers, and code listening on those ports
- Services available on the inside of a firewall
- Code that processes incoming data, email, XML, office documents, and industry-specific custom data exchange formats
- Interfaces, SQL, and Web forms
- An employee with access to sensitive information vulnerable to a social engineering attack

Attack surfaces can be categorized as follows:

- **Network attack surface:** This category refers to vulnerabilities over an enterprise network, wide-area network, or the Internet. Included in this category are network protocol vulnerabilities, such as those used for a denial-of-service attack, disruption of communications links, and various forms of intruder attacks.

- **Software attack surface:** This refers to vulnerabilities in application, utility, or operating system code. A particular focus in this category is Web server software.
- **Human attack surface:** This category refers to vulnerabilities created by personnel or outsiders, such as social engineering, human error, and trusted insiders.

An attack surface analysis is a useful technique for assessing the scale and severity of threats to a system. A systematic analysis of points of vulnerability makes developers and security analysts aware of where security mechanisms are required. Once an attack surface is defined, designers may be able to find ways to make the surface smaller, thus making the task of the adversary more difficult. The attack surface also provides guidance on setting priorities for testing, strengthening security measures, and modifying the service or application.

As illustrated in Figure 1.3, the use of layering, or defense in depth, and attack surface reduction complement each other in mitigating security risk.

Attack Trees

An attack tree is a branching, hierarchical data structure that represents a set of potential techniques for exploiting security vulnerabilities [MAUW05, MOOR01, SCHN99]. The security incident that is the goal of the attack is represented as the root node of the tree, and the ways that an attacker could reach that goal are iteratively and incrementally represented as branches and subnodes of the tree. Each subnode defines a subgoal, and each subgoal may have its own set of further subgoals, and so on. The final nodes on the paths outward from the root, that is, the leaf nodes, represent different ways to initiate an attack. Each node other than a leaf is either an AND-node or an OR-node. To achieve the goal represented by an AND-node, the subgoals represented by all of that node's subnodes must be achieved; and for an OR-node, at least one of the subgoals must be achieved. Branches can be labeled with values representing difficulty, cost, or other attack attributes, so that alternative attacks can be compared.

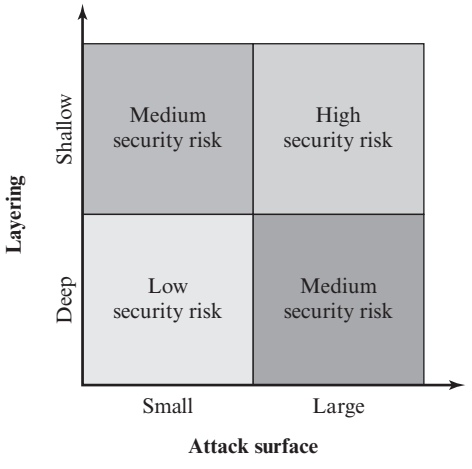

Figure 1.3 Defense in Depth and Attack Surface

The motivation for the use of attack trees is to effectively exploit the information available on attack patterns. Organizations such as CERT publish security advisories that have enabled the development of a body of knowledge about both general attack strategies and specific attack patterns. Security analysts can use the attack tree to document security attacks in a structured form that reveals key vulnerabilities. The attack tree can guide both the design of systems and applications, and the choice and strength of countermeasures.

Figure 1.4, based on a figure in [DIMI07], is an example of an attack tree analysis for an Internet banking authentication application. The root of the tree is the objective of the attacker, which is to compromise a user's account. The shaded boxes on the tree are the leaf nodes, which represent events that comprise the attacks. Note that in this tree, all the nodes other than leaf nodes are OR-nodes. The analysis to generate this tree considered the three components involved in authentication:

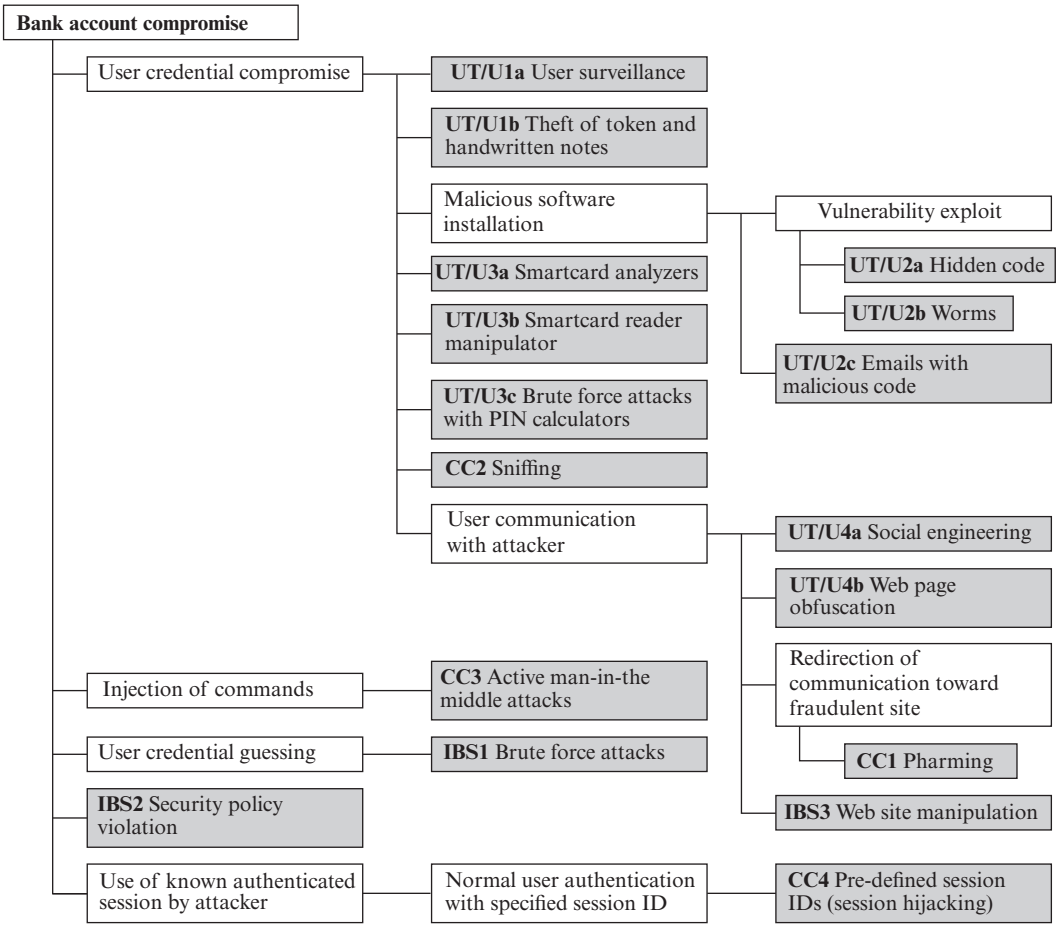

Figure 1.4 An Attack Tree for Internet Banking Authentication

- **User terminal and user (UT/U):** These attacks target the user equipment, including the tokens that may be involved, such as smartcards or other password generators, as well as the actions of the user.
- **Communications channel (CC):** This type of attack focuses on communication links.
- **Internet banking server (IBS):** These types of attacks are offline attacks against the servers that host the Internet banking application.

Five overall attack strategies can be identified, each of which exploits one or more of the three components. The five strategies are as follows:

- **User credential compromise:** This strategy can be used against many elements of the attack surface. There are procedural attacks, such as monitoring a user's action to observe a PIN or other credential, or theft of the user's token or handwritten notes. An adversary may also compromise token information using a variety of token attack tools, such as hacking the smartcard or using a brute force approach to guess the PIN. Another possible strategy is to embed malicious software to compromise the user's login and password. An adversary may also attempt to obtain credential information via the communication channel (sniffing). Finally, an adversary may use various means to engage in communication with the target user, as shown in Figure 1.4.
- **Injection of commands:** In this type of attack, the attacker is able to intercept communication between the UT and the IBS. Various schemes can be used to be able to impersonate the valid user and so gain access to the banking system.
- **User credential guessing:** It is reported in [HILT06] that brute force attacks against some banking authentication schemes are feasible by sending random usernames and passwords. The attack mechanism is based on distributed zombie personal computers, hosting automated programs for username- or password-based calculation.
- **Security policy violation:** For example, violating the bank's security policy in combination with weak access control and logging mechanisms, an employee may cause an internal security incident and expose a customer's account.
- **Use of known authenticated session:** This type of attack persuades or forces the user to connect to the IBS with a preset session ID. Once the user authenticates to the server, the attacker may utilize the known session ID to send packets to the IBS, spoofing the user's identity.

Figure 1.4 provides a thorough view of the different types of attacks on an Internet banking authentication application. Using this tree as a starting point, security analysts can assess the risk of each attack and, using the design principles outlined in the preceding section, design a comprehensive security facility. [DIMO07] provides a good account of the results of this design effort.

1.8 A MODEL FOR NETWORK SECURITY

A model for much of what we will be discussing is captured, in very general terms, in Figure 1.5. A message is to be transferred from one party to another across some sort of Internet service. The two parties, who are the *principals* in this transaction, must cooperate for the exchange to take place. A logical information channel is established by defining a route through the Internet from source to destination and by the cooperative use of communication protocols (e.g., TCP/IP) by the two principals.

Security aspects come into play when it is necessary or desirable to protect the information transmission from an opponent who may present a threat to confidentiality, authenticity, and so on. All the techniques for providing security have two components:

- A security-related transformation on the information to be sent. Examples include the encryption of the message, which scrambles the message so that it is unreadable by the opponent, and the addition of a code based on the contents of the message, which can be used to verify the identity of the sender.
- Some secret information shared by the two principals and, it is hoped, unknown to the opponent. An example is an encryption key used in conjunction with the transformation to scramble the message before transmission and unscramble it on reception.[6]

A trusted third party may be needed to achieve secure transmission. For example, a third party may be responsible for distributing the secret information

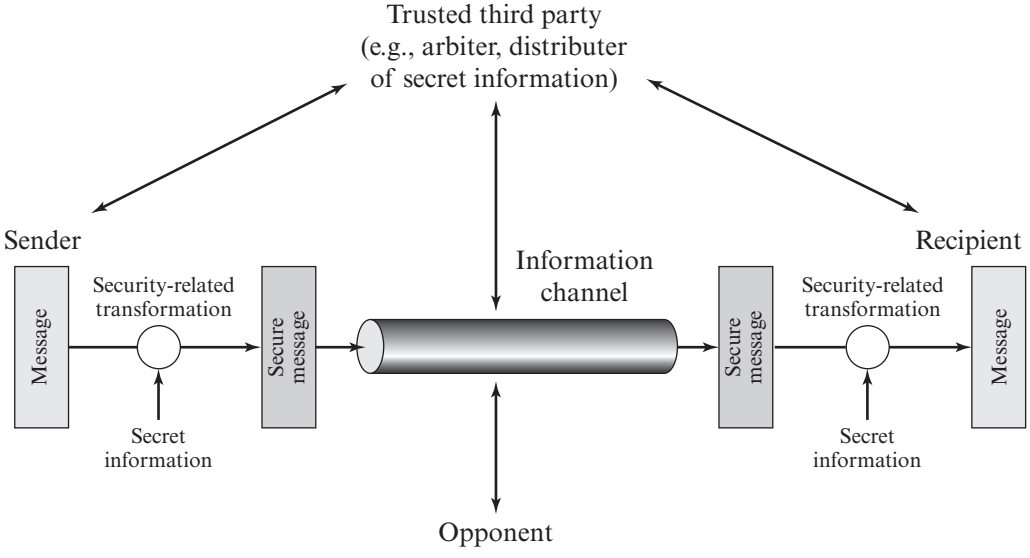

Figure 1.5 Model for Network Security

[6]Part Two discusses a form of encryption, known as a symmetric encryption, in which only one of the two principals needs to have the secret information.

to the two principals while keeping it from any opponent. Or a third party may be needed to arbitrate disputes between the two principals concerning the authenticity of a message transmission.

This general model shows that there are four basic tasks in designing a particular security service:

1. Design an algorithm for performing the security-related transformation. The algorithm should be such that an opponent cannot defeat its purpose.
2. Generate the secret information to be used with the algorithm.
3. Develop methods for the distribution and sharing of the secret information.
4. Specify a protocol to be used by the two principals that makes use of the security algorithm and the secret information to achieve a particular security service.

Parts One through Five of this book concentrate on the types of security mechanisms and services that fit into the model shown in Figure 1.5. However, there are other security-related situations of interest that do not neatly fit this model but are considered in this book. A general model of these other situations is illustrated in Figure 1.6, which reflects a concern for protecting an information system from unwanted access. Most readers are familiar with the concerns caused by the existence of hackers, who attempt to penetrate systems that can be accessed over a network. The hacker can be someone who, with no malign intent, simply gets satisfaction from breaking and entering a computer system. The intruder can be a disgruntled employee who wishes to do damage or a criminal who seeks to exploit computer assets for financial gain (e.g., obtaining credit card numbers or performing illegal money transfers).

Another type of unwanted access is the placement in a computer system of logic that exploits vulnerabilities in the system and that can affect application programs as well as utility programs, such as editors and compilers. Programs can present two kinds of threats:

- **Information access threats:** Intercept or modify data on behalf of users who should not have access to that data.
- **Service threats:** Exploit service flaws in computers to inhibit use by legitimate users.

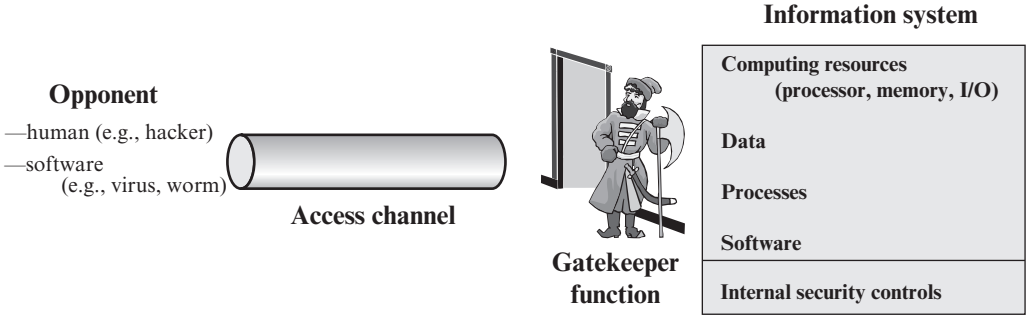

Figure 1.6 Network Access Security Model

Viruses and worms are two examples of software attacks. Such attacks can be introduced into a system by means of a disk that contains the unwanted logic concealed in otherwise useful software. They can also be inserted into a system across a network; this latter mechanism is of more concern in network security.

The security mechanisms needed to cope with unwanted access fall into two broad categories (see Figure 1.6). The first category might be termed a gatekeeper function. It includes password-based login procedures that are designed to deny access to all but authorized users and screening logic that is designed to detect and reject worms, viruses, and other similar attacks. Once either an unwanted user or unwanted software gains access, the second line of defense consists of a variety of internal controls that monitor activity and analyze stored information in an attempt to detect the presence of unwanted intruders. These issues are explored in Part Six.

1.9 STANDARDS

Many of the security techniques and applications described in this book have been specified as standards. Additionally, standards have been developed to cover management practices and the overall architecture of security mechanisms and services. Throughout this book, we describe the most important standards in use or that are being developed for various aspects of cryptography and network security. Various organizations have been involved in the development or promotion of these standards. The most important (in the current context) of these organizations are as follows:

- **National Institute of Standards and Technology:** NIST is a U.S. federal agency that deals with measurement science, standards, and technology related to U.S. government use and to the promotion of U.S. private-sector innovation. Despite its national scope, NIST Federal Information Processing Standards (FIPS) and Special Publications (SP) have a worldwide impact.
- **Internet Society:** ISOC is a professional membership society with worldwide organizational and individual membership. It provides leadership in addressing issues that confront the future of the Internet and is the organization home for the groups responsible for Internet infrastructure standards, including the Internet Engineering Task Force (IETF) and the Internet Architecture Board (IAB). These organizations develop Internet standards and related specifications, all of which are published as Requests for Comments (RFCs).
- **ITU-T:** The International Telecommunication Union (ITU) is an international organization within the United Nations System in which governments and the private sector coordinate global telecom networks and services. The ITU Telecommunication Standardization Sector (ITU-T) is one of the three sectors of the ITU. ITU-T's mission is the development of technical standards covering all fields of telecommunications. ITU-T standards are referred to as Recommendations.

- **ISO:** The International Organization for Standardization (ISO)[7] is a worldwide federation of national standards bodies from more than 140 countries, one from each country. ISO is a nongovernmental organization that promotes the development of standardization and related activities with a view to facilitating the international exchange of goods and services and to developing cooperation in the spheres of intellectual, scientific, technological, and economic activity. ISO's work results in international agreements that are published as International Standards.

A more detailed discussion of these organizations is contained in Appendix D.

1.10 KEY TERMS, REVIEW QUESTIONS, AND PROBLEMS

Key Terms

access control	denial of service	passive attack
active attack	encryption	replay
authentication	integrity	security attacks
authenticity	intruder	security mechanisms
availability	masquerade	security services
data confidentiality	nonrepudiation	traffic analysis
data integrity	OSI security architecture	

Review Questions

1.1 What is the OSI security architecture?
1.2 List and briefly define the three key objectives of computer security.
1.3 List and briefly define categories of passive and active security attacks.
1.4 List and briefly define categories of security services.
1.5 List and briefly define categories of security mechanisms.
1.6 List and briefly define the fundamental security design principles.
1.7 Explain the difference between an attack surface and an attack tree.

Problems

1.1 Consider an automated cash deposit machine in which users provide a card or an account number to deposit cash. Give examples of confidentiality, integrity, and availability requirements associated with the system, and, in each case, indicate the degree of importance of the requirement.
1.2 Repeat Problem 1.1 for a payment gateway system where a user pays for an item using their account via the payment gateway.

[7]ISO is not an acronym (in which case it would be IOS), but it is a word, derived from the Greek, meaning *equal*.

1.3 Consider a financial report publishing system used to produce reports for various organizations.
- **a.** Give an example of a type of publication in which confidentiality of the stored data is the most important requirement.
- **b.** Give an example of a type of publication in which data integrity is the most important requirement.
- **c.** Give an example in which system availability is the most important requirement.

1.4 For each of the following assets, assign a low, moderate, or high impact level for the loss of confidentiality, availability, and integrity, respectively. Justify your answers.
- **a.** A student maintaining a blog to post public information.
- **b.** An examination section of a university that is managing sensitive information about exam papers.
- **c.** An information system in a pathological laboratory maintaining the patient's data.
- **d.** A student information system used for maintaining student data in a university that contains both personal, academic information and routine administrative information (not privacy related). Assess the impact for the two data sets separately and the information system as a whole.
- **e.** A University library contains a library management system which controls the distribution of books amongst the students of various departments. The library management system contains both the student data and the book data. Assess the impact for the two data sets separately and the information system as a whole.

1.5 Draw a matrix similar to Table 1.4 that shows the relationship between security services and attacks.

1.6 Draw a matrix similar to Table 1.4 that shows the relationship between security mechanisms and attacks.

1.7 Develop an attack tree for gaining access to the contents of a physical safe.

1.8 Consider a company whose operations are housed in two buildings on the same property; one building is headquarters, the other building contains network and computer services. The property is physically protected by a fence around the perimeter, and the only entrance to the property is through this fenced perimeter. In addition to the perimeter fence, physical security consists of a guarded front gate. The local networks are split between the Headquarters' LAN and the Network Services' LAN. Internet users connect to the Web server through a firewall. Dial-up users get access to a particular server on the Network Services' LAN. Develop an attack tree in which the root node represents disclosure of proprietary secrets. Include physical, social engineering, and technical attacks. The tree may contain both AND and OR nodes. Develop a tree that has at least 15 leaf nodes.

1.9 Read all of the classic papers cited in the Recommended Reading section for this chapter, available at the Author Web site at WilliamStallings.com/Cryptography. The papers are available at box.com/Crypto7e. Compose a 500–1000 word paper (or 8–12 slide PowerPoint presentation) that summarizes the key concepts that emerge from these papers, emphasizing concepts that are common to most or all of the papers.

CHAPTER 2

INTRODUCTION TO NUMBER THEORY

LEARNING OBJECTIVES

After studying this chapter, you should be able to:

- Understand the concept of divisibility and the division algorithm.
- Understand how to use the Euclidean algorithm to find the greatest common divisor.
- Present an overview of the concepts of modular arithmetic.
- Explain the operation of the extended Euclidean algorithm.
- Discuss key concepts relating to prime numbers.
- Understand Fermat's theorem.
- Understand Euler's theorem.
- Define Euler's totient function.
- Make a presentation on the topic of testing for primality.
- Explain the Chinese remainder theorem.
- Define discrete logarithms.

Number theory is pervasive in cryptographic algorithms. This chapter provides sufficient breadth and depth of coverage of relevant number theory topics for understanding the wide range of applications in cryptography. The reader familiar with these topics can safely skip this chapter.

The first three sections introduce basic concepts from number theory that are needed for understanding finite fields; these include divisibility, the Euclidian algorithm, and modular arithmetic. The reader may study these sections now or wait until ready to tackle Chapter 5 on finite fields.

Sections 2.4 through 2.8 discuss aspects of number theory related to prime numbers and discrete logarithms. These topics are fundamental to the design of asymmetric (public-key) cryptographic algorithms. The reader may study these sections now or wait until ready to read Part Three.

The concepts and techniques of number theory are quite abstract, and it is often difficult to grasp them intuitively without examples. Accordingly, this chapter includes a number of examples, each of which is highlighted in a shaded box.

2.1 DIVISIBILITY AND THE DIVISION ALGORITHM

Divisibility

We say that a nonzero b **divides** a if $a = mb$ for some m, where a, b, and m are integers. That is, b divides a if there is no remainder on division. The notation $b \mid a$ is commonly used to mean b divides a. Also, if $b \mid a$, we say that b is a **divisor** of a.

The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.
$13|182; -5|30; 17|289; -3|33; 17|0$

Subsequently, we will need some simple properties of divisibility for integers, which are as follows:

- If $a|1$, then $a = \pm 1$.
- If $a|b$ and $b|a$, then $a = \pm b$.
- Any $b \neq 0$ divides 0.
- If $a|b$ and $b|c$, then $a|c$:

$$11|66 \text{ and } 66|198 \Rightarrow 11|198$$

- If $b|g$ and $b|h$, then $b|(mg + nh)$ for arbitrary integers m and n.

To see this last point, note that

- If $b|g$, then g is of the form $g = b \times g_1$ for some integer g_1.
- If $b|h$, then h is of the form $h = b \times h_1$ for some integer h_1.

So

$$mg + nh = mbg_1 + nbh_1 = b \times (mg_1 + nh_1)$$

and therefore b divides $mg + nh$.

$b = 7; g = 14; h = 63; m = 3; n = 2$
$7|14$ and $7|63$.
To show $7|(3 \times 14 + 2 \times 63)$,
we have $(3 \times 14 + 2 \times 63) = 7(3 \times 2 + 2 \times 9)$,
and it is obvious that $7|(7(3 \times 2 + 2 \times 9))$.

The Division Algorithm

Given any positive integer n and any nonnegative integer a, if we divide a by n, we get an integer quotient q and an integer remainder r that obey the following relationship:

$$a = qn + r \qquad 0 \leq r < n; q = \lfloor a/n \rfloor \tag{2.1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Equation (2.1) is referred to as the division algorithm.[1]

[1]Equation (2.1) expresses a theorem rather than an algorithm, but by tradition, this is referred to as the division algorithm.

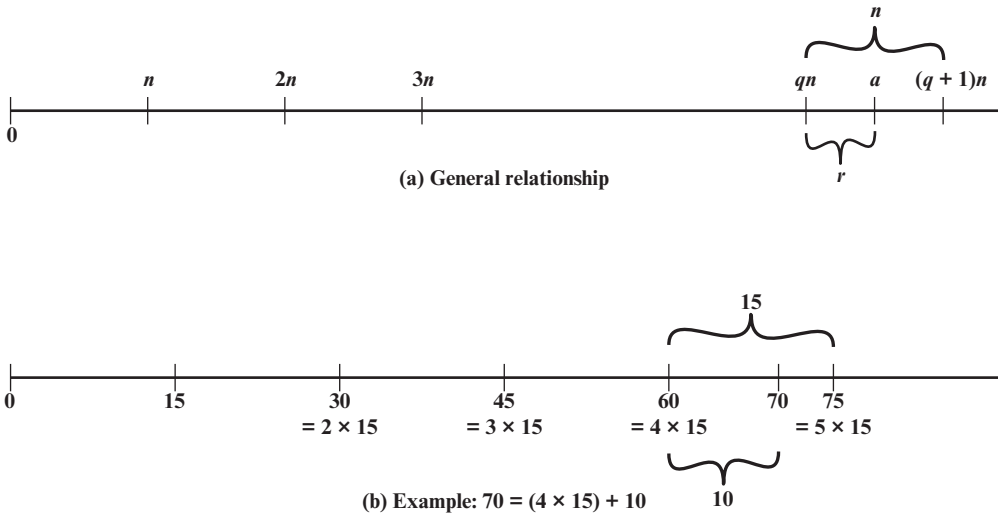

Figure 2.1 The Relationship $a = qn + r; 0 \leq r < n$

Figure 2.1a demonstrates that, given a and positive n, it is always possible to find q and r that satisfy the preceding relationship. Represent the integers on the number line; a will fall somewhere on that line (positive a is shown, a similar demonstration can be made for negative a). Starting at 0, proceed to n, $2n$, up to qn, such that $qn \leq a$ and $(q + 1)n > a$. The distance from qn to a is r, and we have found the unique values of q and r. The remainder r is often referred to as a **residue**.

$a = 11;$	$n = 7;$	$11 = 1 \times 7 + 4;$	$r = 4$	$q = 1$
$a = -11;$	$n = 7;$	$-11 = (-2) \times 7 + 3;$	$r = 3$	$q = -2$

Figure 2.1b provides another example.

2.2 THE EUCLIDEAN ALGORITHM

One of the basic techniques of number theory is the Euclidean algorithm, which is a simple procedure for determining the greatest common divisor of two positive integers. First, we need a simple definition: Two integers are **relatively prime** if and only if their only common positive integer factor is 1.

Greatest Common Divisor

Recall that nonzero b is defined to be a divisor of a if $a = mb$ for some m, where a, b, and m are integers. We will use the notation gcd(a, b) to mean the **greatest common divisor** of a and b. The greatest common divisor of a and b is the largest integer that divides both a and b. We also define $\gcd(0, 0) = 0$.

More formally, the positive integer c is said to be the greatest common divisor of a and b if

1. c is a divisor of a and of b.
2. any divisor of a and b is a divisor of c.

An equivalent definition is the following:

$$\gcd(a, b) = \max[k, \text{such that } k|a \text{ and } k|b]$$

Because we require that the greatest common divisor be positive, $\gcd(a, b) = \gcd(a, -b) = \gcd(-a, b) = \gcd(-a, -b)$. In general, $\gcd(a, b) = \gcd(|a|, |b|)$.

$$\gcd(60, 24) = \gcd(60, -24) = 12$$

Also, because all nonzero integers divide 0, we have $\gcd(a, 0) = |a|$.

We stated that two integers a and b are relatively prime if and only if their only common positive integer factor is 1. This is equivalent to saying that a and b are relatively prime if $\gcd(a, b) = 1$.

8 and 15 are relatively prime because the positive divisors of 8 are 1, 2, 4, and 8, and the positive divisors of 15 are 1, 3, 5, and 15. So 1 is the only integer on both lists.

Finding the Greatest Common Divisor

We now describe an algorithm credited to Euclid for easily finding the greatest common divisor of two integers (Figure 2.2). This algorithm has broad significance in cryptography. The explanation of the algorithm can be broken down into the following points:

1. Suppose we wish to determine the greatest common divisor d of the integers a and b; that is determine $d = \gcd(a, b)$. Because $\gcd(|a|, |b|) = \gcd(a, b)$, there is no harm in assuming $a \geq b > 0$.
2. Dividing a by b and applying the division algorithm, we can state:

$$a = q_1 b + r_1 \qquad 0 \leq r_1 < b \tag{2.2}$$

3. First consider the case in which $r_1 = 0$. Therefore b divides a and clearly no larger number divides both b and a, because that number would be larger than b. So we have $d = \gcd(a, b) = b$.
4. The other possibility from Equation (2.2) is $r_1 \neq 0$. For this case, we can state that $d|r_1$. This is due to the basic properties of divisibility: the relations $d|a$ and $d|b$ together imply that $d|(a - q_1 b)$, which is the same as $d|r_1$.
5. Before proceeding with the Euclidian algorithm, we need to answer the question: What is the $\gcd(b, r_1)$? We know that $d|b$ and $d|r_1$. Now take any arbitrary integer c that divides both b and r_1. Therefore, $c|(q_1 b + r_1) = a$. Because c divides both a and b, we must have $c \leq d$, which is the greatest common divisor of a and b. Therefore $d = \gcd(b, r_1)$.

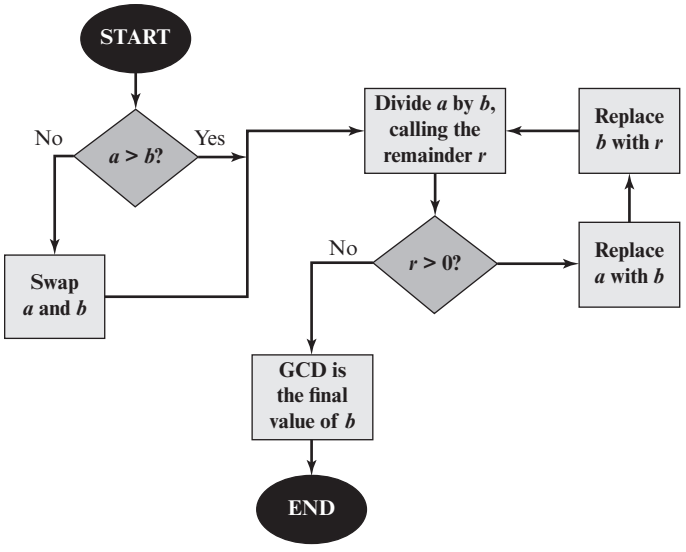

Figure 2.2 Euclidean Algorithm

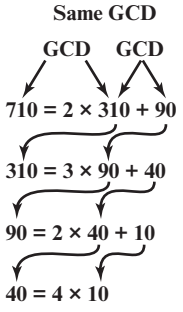

Figure 2.3 Euclidean Algorithm Example: gcd(710, 310)

Let us now return to Equation (2.2) and assume that $r_1 \neq 0$. Because $b > r_1$, we can divide b by r_1 and apply the division algorithm to obtain:

$$b = q_2 r_1 + r_2 \qquad 0 \leq r_2 < r_1$$

As before, if $r_2 = 0$, then $d = r_1$ and if $r_2 \neq 0$, then $d = \gcd(r_1, r_2)$. Note that the remainders form a descending series of nonnegative values and so must terminate when the remainder is zero. This happens, say, at the $(n + 1)$th stage where r_{n-1} is divided by r_n. The result is the following system of equations:

$$\left.\begin{array}{ll} a = q_1 b + r_1 & 0 < r_1 < b \\ b = q_2 r_1 + r_2 & 0 < r_2 < r_1 \\ r_1 = q_3 r_2 + r_3 & 0 < r_3 < r_2 \\ \quad\cdot & \quad\cdot \\ \quad\cdot & \quad\cdot \\ \quad\cdot & \quad\cdot \\ r_{n-2} = q_n r_{n-1} + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} = q_{n+1} r_n + 0 & \\ d = \gcd(a, b) = r_n & \end{array}\right\} \qquad (2.3)$$

At each iteration, we have $d = \gcd(r_i, r_{i+1})$ until finally $d = \gcd(r_n, 0) = r_n$. Thus, we can find the greatest common divisor of two integers by repetitive application of the division algorithm. This scheme is known as the Euclidean algorithm. Figure 2.3 illustrates a simple example.

We have essentially argued from the top down that the final result is the gcd(*a, b*). We can also argue from the bottom up. The first step is to show that r_n divides a and b. It follows from the last division in Equation (2.3) that r_n divides r_{n-1}. The next to last division shows that r_n divides r_{n-2} because it divides both

terms on the right. Successively, one sees that r_n divides all r_i's and finally a and b. It remains to show that r_n is the largest divisor that divides a and b. If we take any arbitrary integer that divides a and b, it must also divide r_1, as explained previously. We can follow the sequence of equations in Equation (2.3) down and show that c must divide all r_i's. Therefore c must divide r_n, so that $r_n = \gcd(a, b)$.

Let us now look at an example with relatively large numbers to see the power of this algorithm:

To find $d = \gcd(a, b) = \gcd(1160718174, 316258250)$		
$a = q_1 b + r_1$	$1160718174 = 3 \times 316258250 + 211943424$	$d = \gcd(316258250, 211943424)$
$b = q_2 r_1 + r_2$	$316258250 = 1 \times 211943424 + 104314826$	$d = \gcd(211943424, 104314826)$
$r_1 = q_3 r_2 + r_3$	$211943424 = 2 \times 104314826 + 3313772$	$d = \gcd(104314826, 3313772)$
$r_2 = q_4 r_3 + r_4$	$104314826 = 31 \times 3313772 + 1587894$	$d = \gcd(3313772, 1587894)$
$r_3 = q_5 r_4 + r_5$	$3313772 = 2 \times 1587894 + 137984$	$d = \gcd(1587894, 137984)$
$r_4 = q_6 r_5 + r_6$	$1587894 = 11 \times 137984 + 70070$	$d = \gcd(137984, 70070)$
$r_5 = q_7 r_6 + r_7$	$137984 = 1 \times 70070 + 67914$	$d = \gcd(70070, 67914)$
$r_6 = q_8 r_7 + r_8$	$70070 = 1 \times 67914 + 2156$	$d = \gcd(67914, 2156)$
$r_7 = q_9 r_8 + r_9$	$67914 = 31 \times 2156 + 1078$	$d = \gcd(2156, 1078)$
$r_8 = q_{10} r_9 + r_{10}$	$2156 = 2 \times 1078 + 0$	$d = \gcd(1078, 0) = 1078$
Therefore, $d = \gcd(1160718174, 316258250) = 1078$		

In this example, we begin by dividing 1160718174 by 316258250, which gives 3 with a remainder of 211943424. Next we take 316258250 and divide it by 211943424. The process continues until we get a remainder of 0, yielding a result of 1078.

It will be helpful in what follows to recast the above computation in tabular form. For every step of the iteration, we have $r_{i-2} = q_i r_{i-1} + r_i$, where r_{i-2} is the dividend, r_{i-1} is the divisor, q_i is the quotient, and r_i is the remainder. Table 2.1 summarizes the results.

Table 2.1 Euclidean Algorithm Example

Dividend	Divisor	Quotient	Remainder
$a = 1160718174$	$b = 316258250$	$q_1 = 3$	$r_1 = 211943424$
$b = 316258250$	$r_1 = 211943434$	$q_2 = 1$	$r_2 = 104314826$
$r_1 = 211943424$	$r_2 = 104314826$	$q_3 = 2$	$r_3 = 3313772$
$r_2 = 104314826$	$r_3 = 3313772$	$q_4 = 31$	$r_4 = 1587894$
$r_3 = 3313772$	$r_4 = 1587894$	$q_5 = 2$	$r_5 = 137984$
$r_4 = 1587894$	$r_5 = 137984$	$q_6 = 11$	$r_6 = 70070$
$r_5 = 137984$	$r_6 = 70070$	$q_7 = 1$	$r_7 = 67914$
$r_6 = 70070$	$r_7 = 67914$	$q_8 = 1$	$r_8 = 2156$
$r_7 = 67914$	$r_8 = 2156$	$q_9 = 31$	$r_9 = 1078$
$r_8 = 2156$	$r_9 = 1078$	$q_{10} = 2$	$r_{10} = 0$

2.3 MODULAR ARITHMETIC

The Modulus

If a is an integer and n is a positive integer, we define $a \bmod n$ to be the remainder when a is divided by n. The integer n is called the **modulus**. Thus, for any integer a, we can rewrite Equation (2.1) as follows:

$$a = qn + r \qquad 0 \leq r < n; q = \lfloor a/n \rfloor$$
$$a = \lfloor a/n \rfloor \times n + (a \bmod n)$$

$$11 \bmod 7 = 4; \qquad -11 \bmod 7 = 3$$

Two integers a and b are said to be **congruent modulo n**, if $(a \bmod n) = (b \bmod n)$. This is written as $a \equiv b \pmod{n}$.[2]

$$73 \equiv 4 \pmod{23}; \qquad 21 \equiv -9 \pmod{10}$$

Note that if $a \equiv 0 \pmod{n}$, then $n \mid a$.

Properties of Congruences

Congruences have the following properties:

1. $a \equiv b \pmod{n}$ if $n \mid (a - b)$.
2. $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$.
3. $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ imply $a \equiv c \pmod{n}$.

To demonstrate the first point, if $n \mid (a - b)$, then $(a - b) = kn$ for some k. So we can write $a = b + kn$. Therefore, $(a \bmod n) =$ (remainder when $b + kn$ is divided by n) = (remainder when b is divided by n) = $(b \bmod n)$.

$23 \equiv 8 \pmod{5}$	because	$23 - 8 = 15 = 5 \times 3$
$-11 \equiv 5 \pmod{8}$	because	$-11 - 5 = -16 = 8 \times (-2)$
$81 \equiv 0 \pmod{27}$	because	$81 - 0 = 81 = 27 \times 3$

The remaining points are as easily proved.

[2]We have just used the operator *mod* in two different ways: first as a **binary operator** that produces a remainder, as in the expression $a \bmod b$; second as a **congruence relation** that shows the equivalence of two integers, as in the expression $a \equiv b \pmod{n}$. See Appendix 2A for a discussion.

Modular Arithmetic Operations

Note that, by definition (Figure 2.1), the (mod n) operator maps all integers into the set of integers $\{0, 1, \ldots, (n - 1)\}$. This suggests the question: Can we perform arithmetic operations within the confines of this set? It turns out that we can; this technique is known as **modular arithmetic**.

Modular arithmetic exhibits the following properties:

1. $[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$
2. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$
3. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

We demonstrate the first property. Define $(a \bmod n) = r_a$ and $(b \bmod n) = r_b$. Then we can write $a = r_a + jn$ for some integer j and $b = r_b + kn$ for some integer k. Then

$$\begin{aligned}(a + b) \bmod n &= (r_a + jn + r_b + kn) \bmod n \\ &= (r_a + r_b + (k + j)n) \bmod n \\ &= (r_a + r_b) \bmod n \\ &= [(a \bmod n) + (b \bmod n)] \bmod n\end{aligned}$$

The remaining properties are proven as easily. Here are examples of the three properties:

$11 \bmod 8 = 3; 15 \bmod 8 = 7$
$[(11 \bmod 8) + (15 \bmod 8)] \bmod 8 = 10 \bmod 8 = 2$
$(11 + 15) \bmod 8 = 26 \bmod 8 = 2$
$[(11 \bmod 8) - (15 \bmod 8)] \bmod 8 = -4 \bmod 8 = 4$
$(11 - 15) \bmod 8 = -4 \bmod 8 = 4$
$[(11 \bmod 8) \times (15 \bmod 8)] \bmod 8 = 21 \bmod 8 = 5$
$(11 \times 15) \bmod 8 = 165 \bmod 8 = 5$

Exponentiation is performed by repeated multiplication, as in ordinary arithmetic.

To find $11^7 \bmod 13$, we can proceed as follows:
$11^2 = 121 \equiv 4 \pmod{13}$
$11^4 = (11^2)^2 \equiv 4^2 \equiv 3 \pmod{13}$
$11^7 = 11 \times 11^2 \times 11^4$
$11^7 \equiv 11 \times 4 \times 3 \equiv 132 \equiv 2 \pmod{13}$

Thus, the rules for ordinary arithmetic involving addition, subtraction, and multiplication carry over into modular arithmetic.

Table 2.2 provides an illustration of modular addition and multiplication modulo 8. Looking at addition, the results are straightforward, and there is a regular pattern to the matrix. Both matrices are symmetric about the main diagonal in conformance to the commutative property of addition and multiplication. As in ordinary addition, there is an additive inverse, or negative, to each integer in modular arithmetic. In this case, the negative of an integer x is the integer y such that $(x + y) \bmod 8 = 0$. To find the additive inverse of an integer in the left-hand column, scan across the corresponding row of the matrix to find the value 0; the integer at the top of that column is the additive inverse; thus, $(2 + 6) \bmod 8 = 0$. Similarly, the entries in the multiplication table are straightforward. In modular arithmetic mod 8, the multiplicative inverse of x is the integer y such that $(x \times y) \bmod 8 = 1 \bmod 8$. Now, to find the multiplicative inverse of an integer from the multiplication table, scan across the matrix in the row for that integer to find the value 1; the integer at the top of that column is the multiplicative inverse; thus, $(3 \times 3) \bmod 8 = 1$. Note that not all integers mod 8 have a multiplicative inverse; more about that later.

Properties of Modular Arithmetic

Define the set Z_n as the set of nonnegative integers less than n:

$$Z_n = \{0, 1, \ldots, (n - 1)\}$$

Table 2.2 Arithmetic Modulo 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

(a) Addition modulo 8

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(b) Multiplication modulo 8

w	$-w$	w^{-1}
0	0	—
1	7	1
2	6	—
3	5	3
4	4	—
5	3	5
6	2	—
7	1	7

(c) Additive and multiplicative inverse modulo 8

This is referred to as the **set of residues**, or **residue classes** (mod n). To be more precise, each integer in Z_n represents a residue class. We can label the residue classes (mod n) as [0], [1], [2], . . . , [$n - 1$], where

$$[r] = \{a: a \text{ is an integer}, a \equiv r \pmod{n}\}$$

The residue classes (mod 4) are
$[0] = \{\ldots, -16, -12, -8, -4, 0, 4, 8, 12, 16, \ldots\}$
$[1] = \{\ldots, -15, -11, -7, -3, 1, 5, 9, 13, 17, \ldots\}$
$[2] = \{\ldots, -14, -10, -6, -2, 2, 6, 10, 14, 18, \ldots\}$
$[3] = \{\ldots, -13, -9, -5, -1, 3, 7, 11, 15, 19, \ldots\}$

Of all the integers in a residue class, the smallest nonnegative integer is the one used to represent the residue class. Finding the smallest nonnegative integer to which k is congruent modulo n is called **reducing *k* modulo *n***.

If we perform modular arithmetic within Z_n, the properties shown in Table 2.3 hold for integers in Z_n. We show in the next section that this implies that Z_n is a commutative ring with a multiplicative identity element.

There is one peculiarity of modular arithmetic that sets it apart from ordinary arithmetic. First, observe that (as in ordinary arithmetic) we can write the following:

$$\textbf{if } (a + b) \equiv (a + c) \pmod{n} \textbf{ then } b \equiv c \pmod{n} \qquad \textbf{(2.4)}$$

$$(5 + 23) \equiv (5 + 7) \pmod{8}; 23 \equiv 7 \pmod{8}$$

Equation (2.4) is consistent with the existence of an additive inverse. Adding the additive inverse of a to both sides of Equation (2.4), we have

$$((-a) + a + b) \equiv ((-a) + a + c) \pmod{n}$$
$$b \equiv c \pmod{n}$$

Table 2.3 Properties of Modular Arithmetic for Integers in Z_n

Property	Expression
Commutative Laws	$(w + x) \bmod n = (x + w) \bmod n$ $(w \times x) \bmod n = (x \times w) \bmod n$
Associative Laws	$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$ $[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$
Distributive Law	$[w \times (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$
Identities	$(0 + w) \bmod n = w \bmod n$ $(1 \times w) \bmod n = w \bmod n$
Additive Inverse ($-w$)	For each $w \in Z_n$, there exists a z such that $w + z \equiv 0 \bmod n$

However, the following statement is true only with the attached condition:

if $(a \times b) \equiv (a \times c) \pmod{n}$ **then** $b \equiv c \pmod{n}$ **if** a is relatively prime to n **(2.5)**

Recall that two integers are **relatively prime** if their only common positive integer factor is 1. Similar to the case of Equation (2.4), we can say that Equation (2.5) is consistent with the existence of a multiplicative inverse. Applying the multiplicative inverse of a to both sides of Equation (2.5), we have

$$((a^{-1})ab) \equiv ((a^{-1})ac) \pmod{n}$$
$$b \equiv c \pmod{n}$$

To see this, consider an example in which the condition of Equation (2.5) does not hold. The integers 6 and 8 are not relatively prime, since they have the common factor 2. We have the following:

$$6 \times 3 = 18 \equiv 2 \pmod{8}$$
$$6 \times 7 = 42 \equiv 2 \pmod{8}$$

Yet $3 \not\equiv 7 \pmod{8}$.

The reason for this strange result is that for any general modulus n, a multiplier a that is applied in turn to the integers 0 through $(n - 1)$ will fail to produce a complete set of residues if a and n have any factors in common.

With $a = 6$ and $n = 8$,

Z_8	0	1	2	3	4	5	6	7
Multiply by 6	0	6	12	18	24	30	36	42
Residues	0	6	4	2	0	6	4	2

Because we do not have a complete set of residues when multiplying by 6, more than one integer in Z_8 maps into the same residue. Specifically, $6 \times 0 \bmod 8 = 6 \times 4 \bmod 8$; $6 \times 1 \bmod 8 = 6 \times 5 \bmod 8$; and so on. Because this is a many-to-one mapping, there is not a unique inverse to the multiply operation.

However, if we take $a = 5$ and $n = 8$, whose only common factor is 1,

Z_8	0	1	2	3	4	5	6	7
Multiply by 5	0	5	10	15	20	25	30	35
Residues	0	5	2	7	4	1	6	3

The line of residues contains all the integers in Z_8, in a different order.

In general, an integer has a multiplicative inverse in Z_n if and only if that integer is relatively prime to n. Table 2.2c shows that the integers 1, 3, 5, and 7 have a multiplicative inverse in Z_8; but 2, 4, and 6 do not.

Euclidean Algorithm Revisited

The Euclidean algorithm can be based on the following theorem: For any integers $a, b,$ with $a \geq b \geq 0$,

$$\gcd(a, b) = \gcd(b, a \bmod b) \tag{2.6}$$

$$\gcd(55, 22) = \gcd(22, 55 \bmod 22) = \gcd(22, 11) = 11$$

To see that Equation (2.6) works, let $d = \gcd(a, b)$. Then, by the definition of gcd, $d|a$ and $d|b$. For any positive integer b, we can express a as

$$a = kb + r \equiv r \,(\text{mod } b)$$
$$a \bmod b = r$$

with k, r integers. Therefore, $(a \bmod b) = a - kb$ for some integer k. But because $d|b$, it also divides kb. We also have $d|a$. Therefore, $d|(a \bmod b)$. This shows that d is a common divisor of b and $(a \bmod b)$. Conversely, if d is a common divisor of b and $(a \bmod b)$, then $d|kb$ and thus $d|[kb + (a \bmod b)]$, which is equivalent to $d|a$. Thus, the set of common divisors of a and b is equal to the set of common divisors of b and $(a \bmod b)$. Therefore, the gcd of one pair is the same as the gcd of the other pair, proving the theorem.

Equation (2.6) can be used repetitively to determine the greatest common divisor.

$$\gcd(18, 12) = \gcd(12, 6) = \gcd(6, 0) = 6$$
$$\gcd(11, 10) = \gcd(10, 1) = \gcd(1, 0) = 1$$

This is the same scheme shown in Equation (2.3), which can be rewritten in the following way.

Euclidean Algorithm	
Calculate	**Which satisfies**
$r_1 = a \bmod b$	$a = q_1 b + r_1$
$r_2 = b \bmod r_1$	$b = q_2 r_1 + r_2$
$r_3 = r_1 \bmod r_2$	$r_1 = q_3 r_2 + r_3$
•	•
•	•
•	•
$r_n = r_{n-2} \bmod r_{n-1}$	$r_{n-2} = q_n r_{n-1} + r_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$	$r_{n-1} = q_{n+1} r_n + 0$ $d = \gcd(a, b) = r_n$

We can define the Euclidean algorithm concisely as the following recursive function.

```
Euclid(a,b)
    if (b=0) then return a;
    else return Euclid(b, a mod b);
```

The Extended Euclidean Algorithm

We now proceed to look at an extension to the Euclidean algorithm that will be important for later computations in the area of finite fields and in encryption algorithms, such as RSA. For given integers a and b, the extended Euclidean algorithm not only calculates the greatest common divisor d but also two additional integers x and y that satisfy the following equation.

$$ax + by = d = \gcd(a, b) \tag{2.7}$$

It should be clear that x and y will have opposite signs. Before examining the algorithm, let us look at some of the values of x and y when $a = 42$ and $b = 30$. Note that $\gcd(42, 30) = 6$. Here is a partial table of values[3] for $42x + 30y$.

x / y	−3	−2	−1	0	1	2	3
−3	−216	−174	−132	−90	−48	−6	36
−2	−186	−144	−102	−60	−18	24	66
−1	−156	−114	−72	−30	12	54	96
0	−126	−84	−42	0	42	84	126
1	−96	−54	−12	30	72	114	156
2	−66	−24	18	60	102	144	186
3	−36	6	48	90	132	174	216

Observe that all of the entries are divisible by 6. This is not surprising, because both 42 and 30 are divisible by 6, so every number of the form $42x + 30y = 6(7x + 5y)$ is a multiple of 6. Note also that $\gcd(42, 30) = 6$ appears in the table. In general, it can be shown that for given integers a and b, the smallest positive value of $ax + by$ is equal to $\gcd(a, b)$.

Now let us show how to extend the Euclidean algorithm to determine (x, y, d) given a and b. We again go through the sequence of divisions indicated in Equation (2.3), and we assume that at each step i we can find integers x_i and y_i that satisfy $r_i = ax_i + by_i$. We end up with the following sequence.

$$\begin{aligned}
a &= q_1 b + r_1 & r_1 &= ax_1 + by_1 \\
b &= q_2 r_1 + r_2 & r_2 &= ax_2 + by_2 \\
r_1 &= q_3 r_2 + r_3 & r_3 &= ax_3 + by_3 \\
&\vdots & &\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n & r_n &= ax_n + by_n \\
r_{n-1} &= q_{n+1} r_n + 0 &&
\end{aligned}$$

[3]This example is taken from [SILV06].

Now, observe that we can rearrange terms to write

$$r_i = r_{i-2} - r_{i-1}q_i \tag{2.8}$$

Also, in rows $i - 1$ and $i - 2$, we find the values

$$r_{i-2} = ax_{i-2} + by_{i-2} \quad \text{and} \quad r_{i-1} = ax_{i-1} + by_{i-1}$$

Substituting into Equation (2.8), we have

$$\begin{aligned} r_i &= (ax_{i-2} + by_{i-2}) - (ax_{i-1} + by_{i-1})q_i \\ &= a(x_{i-2} - q_i x_{i-1}) + b(y_{i-2} - q_i y_{i-1}) \end{aligned}$$

But we have already assumed that $r_i = ax_i + by_i$. Therefore,

$$x_i = x_{i-2} - q_i x_{i-1} \quad \text{and} \quad y_i = y_{i-2} - q_i y_{i-1}$$

We now summarize the calculations:

Extended Euclidean Algorithm			
Calculate	**Which satisfies**	**Calculate**	**Which satisfies**
$r_{-1} = a$		$x_{-1} = 1; y_{-1} = 0$	$a = ax_{-1} + by_{-1}$
$r_0 = b$		$x_0 = 0; y_0 = 1$	$b = ax_0 + by_0$
$r_1 = a \bmod b$ $q_1 = \lfloor a/b \rfloor$	$a = q_1 b + r_1$	$x_1 = x_{-1} - q_1 x_0 = 1$ $y_1 = y_{-1} - q_1 y_0 = -q_1$	$r_1 = ax_1 + by_1$
$r_2 = b \bmod r_1$ $q_2 = \lfloor b/r_1 \rfloor$	$b = q_2 r_1 + r_2$	$x_2 = x_0 - q_2 x_1$ $y_2 = y_0 - q_2 y_1$	$r_2 = ax_2 + by_2$
$r_3 = r_1 \bmod r_2$ $q_3 = \lfloor r_1/r_2 \rfloor$	$r_1 = q_3 r_2 + r_3$	$x_3 = x_1 - q_3 x_2$ $y_3 = y_1 - q_3 y_2$	$r_3 = ax_3 + by_3$
. . .	. . .	. . .	. . .
$r_n = r_{n-2} \bmod r_{n-1}$ $q_n = \lfloor r_{n-2}/r_{n-1} \rfloor$	$r_{n-2} = q_n r_{n-1} + r_n$	$x_n = x_{n-2} - q_n x_{n-1}$ $y_n = y_{n-2} - q_n y_{n-1}$	$r_n = ax_n + by_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$ $q_{n+1} = \lfloor r_{n-1}/r_n \rfloor$	$r_{n-1} = q_{n+1} r_n + 0$		$d = \gcd(a, b) = r_n$ $x = x_n; y = y_n$

We need to make several additional comments here. In each row, we calculate a new remainder r_i based on the remainders of the previous two rows, namely r_{i-1} and r_{i-2}. To start the algorithm, we need values for r_0 and r_{-1}, which are just a and b. It is then straightforward to determine the required values for x_{-1}, y_{-1}, x_0, and y_0.

We know from the original Euclidean algorithm that the process ends with a remainder of zero and that the greatest common divisor of a and b is $d = \gcd(a, b) = r_n$. But we also have determined that $d = r_n = ax_n + by_n$. Therefore, in Equation (2.7), $x = x_n$ and $y = y_n$.

As an example, let us use $a = 1759$ and $b = 550$ and solve for $1759x + 550y = \gcd(1759, 550)$. The results are shown in Table 2.4. Thus, we have $1759 \times (-111) + 550 \times 355 = -195249 + 195250 = 1$.

Table 2.4 Extended Euclidean Algorithm Example

i	r_i	q_i	x_i	y_i
−1	1759		1	0
0	550		0	1
1	109	3	1	−3
2	5	5	−5	16
3	4	21	106	−339
4	1	1	−111	355
5	0	4		

Result: $d = 1; x = -111; y = 355$

2.4 PRIME NUMBERS[4]

A central concern of number theory is the study of prime numbers. Indeed, whole books have been written on the subject (e.g., [CRAN01], [RIBE96]). In this section, we provide an overview relevant to the concerns of this book.

An integer $p > 1$ is a prime number if and only if its only divisors[5] are ± 1 and $\pm p$. **Prime numbers** play a critical role in number theory and in the techniques discussed in this chapter. Table 2.5 shows the primes less than 2000. Note the way the primes are distributed. In particular, note the number of primes in each range of 100 numbers.

Any integer $a > 1$ can be factored in a unique way as

$$a = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_t^{a_t} \tag{2.9}$$

where $p_1 < p_2 < \ldots < p_t$ are prime numbers and where each a_i is a positive integer. This is known as the fundamental theorem of arithmetic; a proof can be found in any text on number theory.

$$\begin{aligned} 91 &= 7 \times 13 \\ 3600 &= 2^4 \times 3^2 \times 5^2 \\ 11011 &= 7 \times 11^2 \times 13 \end{aligned}$$

It is useful for what follows to express this another way. If P is the set of all prime numbers, then any positive integer a can be written uniquely in the following form:

$$a = \prod_{p \in \mathrm{P}} p^{a_p} \quad \text{where each } a_p \geq 0$$

[4]In this section, unless otherwise noted, we deal only with the nonnegative integers. The use of negative integers would introduce no essential differences.

[5]Recall from Section 2.1 that integer a is said to be a divisor of integer b if there is no remainder on division. Equivalently, we say that a divides b.

Table 2.5 Primes Under 2000

2	101	211	307	401	503	601	701	809	907	1009	1103	1201	1301	1409	1511	1601	1709	1801	1901
3	103	223	311	409	509	607	709	811	911	1013	1109	1213	1303	1423	1523	1607	1721	1811	1907
5	107	227	313	419	521	613	719	821	919	1019	1117	1217	1307	1427	1531	1609	1723	1823	1913
7	109	229	317	421	523	617	727	823	929	1021	1123	1223	1319	1429	1543	1613	1733	1831	1931
11	113	233	331	431	541	619	733	827	937	1031	1129	1229	1321	1433	1549	1619	1741	1847	1933
13	127	239	337	433	547	631	739	829	941	1033	1151	1231	1327	1439	1553	1621	1747	1861	1949
17	131	241	347	439	557	641	743	839	947	1039	1153	1237	1361	1447	1559	1627	1753	1867	1951
19	137	251	349	443	563	643	751	853	953	1049	1163	1249	1367	1451	1567	1637	1759	1871	1973
23	139	257	353	449	569	647	757	857	967	1051	1171	1259	1373	1453	1571	1657	1777	1873	1979
29	149	263	359	457	571	653	761	859	971	1061	1181	1277	1381	1459	1579	1663	1783	1877	1987
31	151	269	367	461	577	659	769	863	977	1063	1187	1279	1399	1471	1583	1667	1787	1879	1993
37	157	271	373	463	587	661	773	877	983	1069	1193	1283		1481	1597	1669	1789	1889	1997
41	163	277	379	467	593	673	787	881	991	1087		1289		1483		1693			1999
43	167	281	383	479	599	677	797	883	997	1091		1291		1487		1697			
47	173	283	389	487		683		887		1093		1297		1489		1699			
53	179	293	397	491		691				1097				1493					
59	181			499										1499					
61	191																		
67	193																		
71	197																		
73	199																		
79																			
83																			
89																			
97																			

The right-hand side is the product over all possible prime numbers p; for any particular value of a, most of the exponents a_p will be 0.

The value of any given positive integer can be specified by simply listing all the nonzero exponents in the foregoing formulation.

The integer 12 is represented by $\{a_2 = 2, a_3 = 1\}$.
The integer 18 is represented by $\{a_2 = 1, a_3 = 2\}$.
The integer 91 is represented by $\{a_7 = 1, a_{13} = 1\}$.

Multiplication of two numbers is equivalent to adding the corresponding exponents. Given $a = \prod_{p \in \mathrm{P}} p^{a_p}$, $b = \prod_{p \in \mathrm{P}} p^{b_p}$. Define $k = ab$. We know that the integer k can be expressed as the product of powers of primes: $k = \prod_{p \in \mathrm{P}} p^{k_p}$. It follows that $k_p = a_p + b_p$ for all $p \in \mathrm{P}$.

$k = 12 \times 18 = (2^2 \times 3) \times (2 \times 3^2) = 216$
$k_2 = 2 + 1 = 3; k_3 = 1 + 2 = 3$
$216 = 2^3 \times 3^3 = 8 \times 27$

What does it mean, in terms of the prime factors of a and b, to say that a divides b? Any integer of the form p^n can be divided only by an integer that is of a lesser or equal power of the same prime number, p^j with $j \leq n$. Thus, we can say the following.

Given

$$a = \prod_{p \in \mathrm{P}} p^{a_p}, b = \prod_{p \in \mathrm{P}} p^{b_p}$$

If $a | b$, then $a_p \leq b_p$ for all p.

$a = 12; b = 36; 12 | 36$
$12 = 2^2 \times 3; 36 = 2^2 \times 3^2$
$a_2 = 2 = b_2$
$a_3 = 1 \leq 2 = b_3$
Thus, the inequality $a_p \leq b_p$ is satisfied for all prime numbers.

It is easy to determine the greatest common divisor of two positive integers if we express each integer as the product of primes.

$$300 = 2^2 \times 3^1 \times 5^2$$
$$18 = 2^1 \times 3^2$$
$$\gcd(18,300) = 2^1 \times 3^1 \times 5^0 = 6$$

The following relationship always holds:

$$\text{If } k = \gcd(a, b), \text{ then } k_p = \min(a_p, b_p) \text{ for all } p.$$

Determining the prime factors of a large number is no easy task, so the preceding relationship does not directly lead to a practical method of calculating the greatest common divisor.

2.5 FERMAT'S AND EULER'S THEOREMS

Two theorems that play important roles in public-key cryptography are Fermat's theorem and Euler's theorem.

Fermat's Theorem[6]

Fermat's theorem states the following: If p is prime and a is a positive integer not divisible by p, then

$$a^{p-1} \equiv 1 \pmod{p} \tag{2.10}$$

Proof: Consider the set of positive integers less than p: $\{1, 2, \ldots, p-1\}$ and multiply each element by a, modulo p, to get the set $X = \{a \bmod p, 2a \bmod p, \ldots, (p-1)a \bmod p\}$. None of the elements of X is equal to zero because p does not divide a. Furthermore, no two of the integers in X are equal. To see this, assume that $ja \equiv ka \pmod{p}$, where $1 \le j < k \le p-1$. Because a is relatively prime[7] to p, we can eliminate a from both sides of the equation [see Equation (2.3)] resulting in $j \equiv k \pmod{p}$. This last equality is impossible, because j and k are both positive integers less than p. Therefore, we know that the $(p-1)$ elements of X are all positive integers with no two elements equal. We can conclude the X consists of the set of integers $\{1, 2, \ldots, p-1\}$ in some order. Multiplying the numbers in both sets (p and X) and taking the result mod p yields

$$a \times 2a \times \cdots \times (p-1)a \equiv [(1 \times 2 \times \cdots \times (p-1)] \pmod{p}$$
$$a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$$

We can cancel the $(p-1)!$ term because it is relatively prime to p [see Equation (2.5)]. This yields Equation (2.10), which completes the proof.

---

[6] This is sometimes referred to as Fermat's little theorem.

[7] Recall from Section 2.2 that two numbers are relatively prime if they have no prime factors in common; that is, their only common divisor is 1. This is equivalent to saying that two numbers are relatively prime if their greatest common divisor is 1.

$a = 7, p = 19$
$7^2 = 49 \equiv 11 \pmod{19}$
$7^4 \equiv 121 \equiv 7 \pmod{19}$
$7^8 \equiv 49 \equiv 11 \pmod{19}$
$7^{16} \equiv 121 \equiv 7 \pmod{19}$
$a^{p-1} = 7^{18} = 7^{16} \times 7^2 \equiv 7 \times 11 \equiv 1 \pmod{19}$

An alternative form of Fermat's theorem is also useful: If p is prime and a is a positive integer, then

$$a^p \equiv a \pmod{p} \tag{2.11}$$

Note that the first form of the theorem [Equation (2.10)] requires that a be relatively prime to p, but this form does not.

$p = 5, a = 3$	$a^p = 3^5 = 243 \equiv 3 \pmod{5} = a \pmod{p}$
$p = 5, a = 10$	$a^p = 10^5 = 100000 \equiv 10 \pmod{5} \equiv 0 \pmod{5} = a \pmod{p}$

Euler's Totient Function

Before presenting Euler's theorem, we need to introduce an important quantity in number theory, referred to as **Euler's totient function**. This function, written $\phi(n)$, is defined as the number of positive integers less than n and relatively prime to n. By convention, $\phi(1) = 1$.

Determine $\phi(37)$ and $\phi(35)$.

Because 37 is prime, all of the positive integers from 1 through 36 are relatively prime to 37. Thus $\phi(37) = 36$.

To determine $\phi(35)$, we list all of the positive integers less than 35 that are relatively prime to it:

1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 16, 17, 18
19, 22, 23, 24, 26, 27, 29, 31, 32, 33, 34

There are 24 numbers on the list, so $\phi(35) = 24$.

Table 2.6 lists the first 30 values of $\phi(n)$. The value $\phi(1)$ is without meaning but is defined to have the value 1.

It should be clear that, for a prime number p,

$$\phi(p) = p - 1$$

Now suppose that we have two prime numbers p and q with $p \neq q$. Then we can show that, for $n = pq$,

Table 2.6 Some Values of Euler's Totient Function $\phi(n)$

n	$\phi(n)$	n	$\phi(n)$	n	$\phi(n)$
1	1	11	10	21	12
2	1	12	4	22	10
3	2	13	12	23	22
4	2	14	6	24	8
5	4	15	8	25	20
6	2	16	8	26	12
7	6	17	16	27	18
8	4	18	6	28	12
9	6	19	18	29	28
10	4	20	8	30	8

$$\phi(n) = \phi(pq) = \phi(p) \times \phi(q) = (p - 1) \times (q - 1)$$

To see that $\phi(n) = \phi(p) \times \phi(q)$, consider that the set of positive integers less than n is the set $\{1, \ldots, (pq - 1)\}$. The integers in this set that are not relatively prime to n are the set $\{p, 2p, \ldots, (q - 1)p\}$ and the set $\{q, 2q, \ldots, (p - 1)q\}$. To see this, consider that any integer that divides n must divide either of the prime numbers p or q. Therefore, any integer that does not contain either p or q as a factor is relatively prime to n. Further note that the two sets just listed are non-overlapping: Because p and q are prime, we can state that none of the integers in the first set can be written as a multiple of q, and none of the integers in the second set can be written as a multiple of p. Thus the total number of unique integers in the two sets is $(q - 1) + (p - 1)$. Accordingly,

$$\begin{aligned}\phi(n) &= (pq - 1) - [(q - 1) + (p - 1)] \\ &= pq - (p + q) + 1 \\ &= (p - 1) \times (q - 1) \\ &= \phi(p) \times \phi(q)\end{aligned}$$

$\phi(21) = \phi(3) \times \phi(7) = (3 - 1) \times (7 - 1) = 2 \times 6 = 12$
where the 12 integers are $\{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$.

Euler's Theorem

Euler's theorem states that for every a and n that are relatively prime:

$$a^{\phi(n)} \equiv 1 (\text{mod } n) \tag{2.12}$$

Proof: Equation (2.12) is true if n is prime, because in that case, $\phi(n) = (n - 1)$ and Fermat's theorem holds. However, it also holds for any integer n. Recall that

$\phi(n)$ is the number of positive integers less than n that are relatively prime to n. Consider the set of such integers, labeled as

$$R = \{x_1, x_2, \ldots, x_{\phi(n)}\}$$

That is, each element x_i of R is a unique positive integer less than n with $\gcd(x_i, n) = 1$. Now multiply each element by a, modulo n:

$$S = \{(ax_1 \bmod n), (ax_2 \bmod n), \ldots, (ax_{\phi(n)} \bmod n)\}$$

The set S is a permutation[8] of R , by the following line of reasoning:

1. Because a is relatively prime to n and x_i is relatively prime to n, ax_i must also be relatively prime to n. Thus, all the members of S are integers that are less than n and that are relatively prime to n.
2. There are no duplicates in S. Refer to Equation (2.5). If $ax_i \bmod n = ax_j \bmod n$, then $x_i = x_j$.

Therefore,

$$\begin{aligned}
\prod_{i=1}^{\phi(n)} (ax_i \bmod n) &= \prod_{i=1}^{\phi(n)} x_i \\
\prod_{i=1}^{\phi(n)} ax_i &\equiv \prod_{i=1}^{\phi(n)} x_i \pmod n \\
a^{\phi(n)} \times \left[\prod_{i=1}^{\phi(n)} x_i\right] &\equiv \prod_{i=1}^{\phi(n)} x_i \pmod n \\
a^{\phi(n)} &\equiv 1 \pmod n
\end{aligned}$$

which completes the proof. This is the same line of reasoning applied to the proof of Fermat's theorem.

$a = 3; n = 10; \phi(10) = 4;$	$a^{\phi(n)} = 3^4 = 81 = 1 \pmod{10} = 1 \pmod n$
$a = 2; n = 11; \phi(11) = 10;$	$a^{\phi(n)} = 2^{10} = 1024 = 1 \pmod{11} = 1 \pmod n$

As is the case for Fermat's theorem, an alternative form of the theorem is also useful:

$$a^{\phi(n)+1} \equiv a \pmod n \tag{2.13}$$

Again, similar to the case with Fermat's theorem, the first form of Euler's theorem [Equation (2.12)] requires that a be relatively prime to n, but this form does not.

[8]A permutation of a finite set of elements S is an ordered sequence of all the elements of S, with each element appearing exactly once.

2.6 TESTING FOR PRIMALITY

For many cryptographic algorithms, it is necessary to select one or more very large prime numbers at random. Thus, we are faced with the task of determining whether a given large number is prime. There is no simple yet efficient means of accomplishing this task.

In this section, we present one attractive and popular algorithm. You may be surprised to learn that this algorithm yields a number that is not necessarily a prime. However, the algorithm can yield a number that is almost certainly a prime. This will be explained presently. We also make reference to a deterministic algorithm for finding primes. The section closes with a discussion concerning the distribution of primes.

Miller–Rabin Algorithm[9]

The algorithm due to Miller and Rabin [MILL75, RABI80] is typically used to test a large number for primality. Before explaining the algorithm, we need some background. First, any positive odd integer $n \geq 3$ can be expressed as

$$n - 1 = 2^k q \qquad \text{with } k > 0, q \text{ odd}$$

To see this, note that $n - 1$ is an even integer. Then, divide $(n - 1)$ by 2 until the result is an odd number q, for a total of k divisions. If n is expressed as a binary number, then the result is achieved by shifting the number to the right until the rightmost digit is a 1, for a total of k shifts. We now develop two properties of prime numbers that we will need.

Two Properties of Prime Numbers The **first property** is stated as follows: If p is prime and a is a positive integer less than p, then $a^2 \bmod p = 1$ if and only if either $a \bmod p = 1$ or $a \bmod p = -1 \bmod p = p - 1$. By the rules of modular arithmetic $(a \bmod p)\ (a \bmod p) = a^2 \bmod p$. Thus, if either $a \bmod p = 1$ or $a \bmod p = -1$, then $a^2 \bmod p = 1$. Conversely, if $a^2 \bmod p = 1$, then $(a \bmod p)^2 = 1$, which is true only for $a \bmod p = 1$ or $a \bmod p = -1$.

The **second property** is stated as follows: Let p be a prime number greater than 2. We can then write $p - 1 = 2^k q$ with $k > 0$, q odd. Let a be any integer in the range $1 < a < p - 1$. Then one of the two following conditions is true.

1. a^q is congruent to 1 modulo p. That is, $a^q \bmod p = 1$, or equivalently, $a^q \equiv 1 (\bmod p)$.
2. One of the numbers $a^q, a^{2q}, a^{4q}, \ldots, a^{2^{k-1}q}$ is congruent to -1 modulo p. That is, there is some number j in the range $(1 \leq j \leq k)$ such that $a^{2^{j-1}q} \bmod p = -1 \bmod p = p - 1$ or equivalently, $a^{2^{j-1}q} \equiv -1 (\bmod p)$.

Proof: Fermat's theorem [Equation (2.10)] states that $a^{n-1} \equiv 1 (\bmod n)$ if n is prime. We have $p - 1 = 2^k q$. Thus, we know that $a^{p-1} \bmod p = a^{2^k q} \bmod p = 1$. Thus, if we look at the sequence of numbers

$$a^q \bmod p, a^{2q} \bmod p, a^{4q} \bmod p, \ldots, a^{2^{k-1}q} \bmod p, a^{2^k q} \bmod p \qquad \textbf{(2.14)}$$

[9] Also referred to in the literature as the Rabin-Miller algorithm, or the Rabin-Miller test, or the Miller–Rabin test.

we know that the last number in the list has value 1. Further, each number in the list is the square of the previous number. Therefore, one of the following possibilities must be true.

1. The first number on the list, and therefore all subsequent numbers on the list, equals 1.
2. Some number on the list does not equal 1, but its square mod p does equal 1. By virtue of the first property of prime numbers defined above, we know that the only number that satisfies this condition is $p - 1$. So, in this case, the list contains an element equal to $p - 1$.

This completes the proof.

Details of the Algorithm These considerations lead to the conclusion that, if n is prime, then either the first element in the list of residues, or remainders, $(a^q, a^{2q}, \ldots, a^{2^{k-1}q}, a^{2^k q})$ modulo n equals 1; or some element in the list equals $(n - 1)$; otherwise n is composite (i.e., not a prime). On the other hand, if the condition is met, that does not necessarily mean that n is prime. For example, if $n = 2047 = 23 \times 89$, then $n - 1 = 2 \times 1023$. We compute $2^{1023} \bmod 2047 = 1$, so that 2047 meets the condition but is not prime.

We can use the preceding property to devise a test for primality. The procedure TEST takes a candidate integer n as input and returns the result `composite` if n is definitely not a prime, and the result `inconclusive` if n may or may not be a prime.

```
TEST (n)
1. Find integers k, q, with k > 0, q odd, so that
   (n - 1 = 2k q);
2. Select a random integer a, 1 < a < n - 1;
3. if aq mod n = 1 then return("inconclusive");
4. for j = 0 to k - 1 do
5.   if a^(2^j q) mod n = n - 1 then return("inconclusive");
6. return("composite");
```

Let us apply the test to the prime number $n = 29$. We have $(n - 1) = 28 = 2^2(7) = 2^k q$. First, let us try $a = 10$. We compute $10^7 \bmod 29 = 17$, which is neither 1 nor 28, so we continue the test. The next calculation finds that $(10^7)^2 \bmod 29 = 28$, and the test returns `inconclusive` (i.e., 29 may be prime). Let's try again with $a = 2$. We have the following calculations: $2^7 \bmod 29 = 12$; $2^{14} \bmod 29 = 28$; and the test again returns `inconclusive`. If we perform the test for all integers a in the range 1 through 28, we get the same `inconclusive` result, which is compatible with n being a prime number.

Now let us apply the test to the composite number $n = 13 \times 17 = 221$. Then $(n - 1) = 220 = 2^2(55) = 2^k q$. Let us try $a = 5$. Then we have $5^{55} \bmod 221 = 112$, which is neither 1 nor $220 (5^{55})^2 \bmod 221 = 168$. Because we have used all values of j (i.e., $j = 0$ and $j = 1$) in line 4 of the TEST algorithm, the test returns `composite`, indicating that 221 is definitely a composite number. But suppose we had selected $a = 21$. Then we have $21^{55} \bmod 221 = 200$; $(21^{55})^2 \bmod 221 = 220$; and the test returns `inconclusive`, indicating that 221 may be prime. In fact, of the 218 integers from 2 through 219, four of these will return an inconclusive result, namely 21, 47, 174, and 200.

Repeated Use of the Miller–Rabin Algorithm How can we use the Miller–Rabin algorithm to determine with a high degree of confidence whether or not an integer is prime? It can be shown [KNUT98] that given an odd number n that is not prime and a randomly chosen integer, a with $1 < a < n - 1$, the probability that TEST will return `inconclusive` (i.e., fail to detect that n is not prime) is less than 1/4. Thus, if t different values of a are chosen, the probability that all of them will pass TEST (return inconclusive) for n is less than $(1/4)^t$. For example, for $t = 10$, the probability that a nonprime number will pass all ten tests is less than 10^{-6}. Thus, for a sufficiently large value of t , we can be confident that n is prime if Miller's test always returns `inconclusive`.

This gives us a basis for determining whether an odd integer n is prime with a reasonable degree of confidence. The procedure is as follows: Repeatedly invoke TEST (n) using randomly chosen values for a. If, at any point, TEST returns `composite`, then n is determined to be nonprime. If TEST continues to return `inconclusive` for t tests, then for a sufficiently large value of t, assume that n is prime.

A Deterministic Primality Algorithm

Prior to 2002, there was no known method of efficiently proving the primality of very large numbers. All of the algorithms in use, including the most popular (Miller–Rabin), produced a probabilistic result. In 2002 (announced in 2002, published in 2004), Agrawal, Kayal, and Saxena [AGRA04] developed a relatively simple deterministic algorithm that efficiently determines whether a given large number is a prime. The algorithm, known as the AKS algorithm, does not appear to be as efficient as the Miller–Rabin algorithm. Thus far, it has not supplanted this older, probabilistic technique.

Distribution of Primes

It is worth noting how many numbers are likely to be rejected before a prime number is found using the Miller–Rabin test, or any other test for primality. A result from number theory, known as the prime number theorem, states that the primes near n are spaced on the average one every ln (n) integers. Thus, on average, one would have to test on the order of ln(n) integers before a prime is found. Because all even integers can be immediately rejected, the correct figure is 0.5 ln(n). For example, if a prime on the order of magnitude of 2^{200} were sought, then about $0.5 \ln(2^{200}) = 69$ trials would be needed to find a prime. However, this figure is just an average. In some places along the number line, primes are closely packed, and in other places there are large gaps.

The two consecutive odd integers 1,000,000,000,061 and 1,000,000,000,063 are both prime. On the other hand, 1001! + 2, 1001! + 3, . . . , 1001! + 1000, 1001! + 1001 is a sequence of 1000 consecutive composite integers.

2.7 THE CHINESE REMAINDER THEOREM

One of the most useful results of number theory is the **Chinese remainder theorem** (CRT).[10] In essence, the CRT says it is possible to reconstruct integers in a certain range from their residues modulo a set of pairwise relatively prime moduli.

> The 10 integers in Z_{10}, that is the integers 0 through 9, can be reconstructed from their two residues modulo 2 and 5 (the relatively prime factors of 10). Say the known residues of a decimal digit x are $r_2 = 0$ and $r_5 = 3$; that is, $x \bmod 2 = 0$ and $x \bmod 5 = 3$. Therefore, x is an even integer in Z_{10} whose remainder, on division by 5, is 3. The unique solution is $x = 8$.

The CRT can be stated in several ways. We present here a formulation that is most useful from the point of view of this text. An alternative formulation is explored in Problem 2.33. Let

$$M = \prod_{i=1}^{k} m_i$$

where the m_i are pairwise relatively prime; that is, $\gcd(m_i, m_j) = 1$ for $1 \leq i, j \leq k$, and $i \neq j$. We can represent any integer A in Z_M by a k-tuple whose elements are in Z_{m_i} using the following correspondence:

$$A \leftrightarrow (a_1, a_2, \ldots, a_k) \tag{2.15}$$

where $A \in Z_M$, $a_i \in Z_{m_i}$, and $a_i = A \bmod m_i$ for $1 \leq i \leq k$. The CRT makes two assertions.

1. The mapping of Equation (2.15) is a one-to-one correspondence (called a **bijection**) between Z_M and the Cartesian product $Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_k}$. That is, for every integer A such that $0 \leq A < M$, there is a unique k-tuple $(a_1, a_2, \ldots, a_k)$ with $0 \leq a_i < m_i$ that represents it, and for every such k-tuple $(a_1, a_2, \ldots, a_k)$, there is a unique integer A in Z_M.
2. Operations performed on the elements of Z_M can be equivalently performed on the corresponding k-tuples by performing the operation independently in each coordinate position in the appropriate system.

Let us demonstrate the **first assertion**. The transformation from A to $(a_1, a_2, \ldots, a_k)$, is obviously unique; that is, each a_i is uniquely calculated as $a_i = A \bmod m_i$. Computing A from $(a_1, a_2, \ldots, a_k)$ can be done as follows. Let

[10] The CRT is so called because it is believed to have been discovered by the Chinese mathematician Sun-Tsu in around 100 A.D.

$M_i = M/m_i$ for $1 \leq i \leq k$. Note that $M_i = m_1 \times m_2 \times \ldots \times m_{i-1} \times m_{i+1} \times \ldots \times m_k$, so that $M_i \equiv 0 \pmod{m_j}$ for all $j \neq i$. Then let

$$c_i = M_i \times (M_i^{-1} \bmod m_i) \qquad \text{for } 1 \leq i \leq k \tag{2.16}$$

By the definition of M_i, it is relatively prime to m_i and therefore has a unique multiplicative inverse mod m_i. So Equation (2.16) is well defined and produces a unique value c_i. We can now compute

$$A \equiv \left(\sum_{i=1}^{k} a_i c_i \right) (\bmod M) \tag{2.17}$$

To show that the value of A produced by Equation (2.17) is correct, we must show that $a_i = A \bmod m_i$ for $1 \leq i \leq k$. Note that $c_j \equiv M_j \equiv 0 \pmod{m_i}$ if $j \neq i$, and that $c_i \equiv 1 \pmod{m_i}$. It follows that $a_i = A \bmod m_i$.

The **second assertion** of the CRT, concerning arithmetic operations, follows from the rules for modular arithmetic. That is, the second assertion can be stated as follows: If

$$A \leftrightarrow (a_1, a_2, \ldots, a_k)$$
$$B \leftrightarrow (b_1, b_2, \ldots, b_k)$$

then

$$(A + B) \bmod M \leftrightarrow ((a_1 + b_1) \bmod m_1, \ldots, (a_k + b_k) \bmod m_k)$$
$$(A - B) \bmod M \leftrightarrow ((a_1 - b_1) \bmod m_1, \ldots, (a_k - b_k) \bmod m_k)$$
$$(A \times B) \bmod M \leftrightarrow ((a_1 \times b_1) \bmod m_1, \ldots, (a_k \times b_k) \bmod m_k)$$

One of the useful features of the Chinese remainder theorem is that it provides a way to manipulate (potentially very large) numbers mod M in terms of tuples of smaller numbers. This can be useful when M is 150 digits or more. However, note that it is necessary to know beforehand the factorization of M.

To represent 973 mod 1813 as a pair of numbers mod 37 and 49, define

$$m_1 = 37$$
$$m_2 = 49$$
$$M = 1813$$
$$A = 973$$

We also have $M_1 = 49$ and $M_2 = 37$. Using the extended Euclidean algorithm, we compute $M_1^{-1} = 34 \bmod m_1$ and $M_2^{-1} = 4 \bmod m_2$. (Note that we only need to compute each M_i and each M_i^{-1} once.) Taking residues modulo 37 and 49, our representation of 973 is (11, 42), because 973 mod 37 = 11 and 973 mod 49 = 42.

Now suppose we want to add 678 to 973. What do we do to (11, 42)? First we compute (678) $\leftrightarrow$ (678 mod 37, 678 mod 49) = (12, 41). Then we add the tuples element-wise and reduce (11 + 12 mod 37, 42 + 41 mod 49) = (23, 34). To verify that this has the correct effect, we compute

$$
\begin{aligned}
(23, 34) &\leftrightarrow a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} \bmod M \\
&= [(23)(49)(34) + (34)(37)(4)] \bmod 1813 \\
&= 43350 \bmod 1813 \\
&= 1651
\end{aligned}
$$

and check that it is equal to $(973 + 678) \bmod 1813 = 1651$. Remember that in the above derivation, M_i^{-1} is the multiplicative inverse of M_1 modulo m_1 and M_2^{-1} is the multiplicative inverse of M_2 modulo m_2.

Suppose we want to multiply 1651 (mod 1813) by 73. We multiply (23, 34) by 73 and reduce to get $(23 \times 73 \bmod 37, 34 \times 73 \bmod 49) = (14, 32)$. It is easily verified that

$$
\begin{aligned}
(14, 32) &\leftrightarrow [(14)(49)(34) + (32)(37)(4)] \bmod 1813 \\
&= 865 \\
&= 1651 \times 73 \bmod 1813
\end{aligned}
$$

2.8 DISCRETE LOGARITHMS

Discrete logarithms are fundamental to a number of public-key algorithms, including Diffie–Hellman key exchange and the digital signature algorithm (DSA). This section provides a brief overview of discrete logarithms. For the interested reader, more detailed developments of this topic can be found in [ORE67] and [LEVE90].

The Powers of an Integer, Modulo *n*

Recall from Euler's theorem [Equation (2.12)] that, for every a and n that are relatively prime,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

where $\phi(n)$, Euler's totient function, is the number of positive integers less than n and relatively prime to n. Now consider the more general expression:

$$a^m \equiv 1 \pmod{n} \tag{2.18}$$

If a and n are relatively prime, then there is at least one integer m that satisfies Equation (2.18), namely, $m = \phi(n)$. The least positive exponent m for which Equation (2.18) holds is referred to in several ways:

- The order of a (mod n)
- The exponent to which a belongs (mod n)
- The length of the period generated by a

To see this last point, consider the powers of 7, modulo 19:

$$\begin{aligned} 7^1 &\equiv && 7 \pmod{19} \\ 7^2 &= 49 = 2 \times 19 + 11 && \equiv 11 \pmod{19} \\ 7^3 &= 343 = 18 \times 19 + 1 && \equiv 1 \pmod{19} \\ 7^4 &= 2401 = 126 \times 19 + 7 && \equiv 7 \pmod{19} \\ 7^5 &= 16807 = 884 \times 19 + 11 && \equiv 11 \pmod{19} \end{aligned}$$

There is no point in continuing because the sequence is repeating. This can be proven by noting that $7^3 \equiv 1 (\text{mod } 19)$, and therefore, $7^{3+j} \equiv 7^3 7^j \equiv 7^j (\text{mod } 19)$, and hence, any two powers of 7 whose exponents differ by 3 (or a multiple of 3) are congruent to each other (mod 19). In other words, the sequence is periodic, and the length of the period is the smallest positive exponent m such that $7^m \equiv 1 (\text{mod } 19)$.

Table 2.7 shows all the powers of a, modulo 19 for all positive $a < 19$. The length of the sequence for each base value is indicated by shading. Note the following:

1. All sequences end in 1. This is consistent with the reasoning of the preceding few paragraphs.
2. The length of a sequence divides $\phi(19) = 18$. That is, an integral number of sequences occur in each row of the table.
3. Some of the sequences are of length 18. In this case, it is said that the base integer a generates (via powers) the set of nonzero integers modulo 19. Each such integer is called a primitive root of the modulus 19.

More generally, we can say that the highest possible exponent to which a number can belong (mod n) is $\phi(n)$. If a number is of this order, it is referred to as a **primitive root** of n. The importance of this notion is that if a is a primitive root of n, then its powers

$$a, a^2, \ \ldots \ , a^{\phi(n)}$$

are distinct (mod n) and are all relatively prime to n. In particular, for a prime number p, if a is a primitive root of p, then

$$a, a^2, \ \ldots \ , a^{p-1}$$

are distinct (mod p). For the prime number 19, its primitive roots are 2, 3, 10, 13, 14, and 15.

Not all integers have primitive roots. In fact, the only integers with primitive roots are those of the form 2, 4, p^α, and $2p^\alpha$, where p is any odd prime and α is a positive integer. The proof is not simple but can be found in many number theory books, including [ORE76].

Table 2.7 Powers of Integers, Modulo 19

a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}	a^{18}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
3	9	8	5	15	7	2	6	18	16	10	11	14	4	12	17	13	1
4	16	7	9	17	11	6	5	1	4	16	7	9	17	11	6	5	1
5	6	11	17	9	7	16	4	1	5	6	11	17	9	7	16	4	1
6	17	7	4	5	11	9	16	1	6	17	7	4	5	11	9	16	1
7	11	1	7	11	1	7	11	1	7	11	1	7	11	1	7	11	1
8	7	18	11	12	1	8	7	18	11	12	1	8	7	18	11	12	1
9	5	7	6	16	11	4	17	1	9	5	7	6	16	11	4	17	1
10	5	12	6	3	11	15	17	18	9	14	7	13	16	8	4	2	1
11	7	1	11	7	1	11	7	1	11	7	1	11	7	1	11	7	1
12	11	18	7	8	1	12	11	18	7	8	1	12	11	18	7	8	1
13	17	12	4	14	11	10	16	18	6	2	7	15	5	8	9	3	1
14	6	8	17	10	7	3	4	18	5	13	11	2	9	12	16	15	1
15	16	12	9	2	11	13	5	18	4	3	7	10	17	8	6	14	1
16	9	11	5	4	7	17	6	1	16	9	11	5	4	7	17	6	1
17	4	11	16	6	7	5	9	1	17	4	11	16	6	7	5	9	1
18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1

Logarithms for Modular Arithmetic

With ordinary positive real numbers, the logarithm function is the inverse of exponentiation. An analogous function exists for modular arithmetic.

Let us briefly review the properties of ordinary logarithms. The logarithm of a number is defined to be the power to which some positive base (except 1) must be raised in order to equal the number. That is, for base x and for a value y,

$$y = x^{\log_x(y)}$$

The properties of logarithms include

$$\log_x(1) = 0$$
$$\log_x(x) = 1$$

$$\log_x(yz) = log_x(y) + \log_x(z) \tag{2.19}$$

$$\log_x(y^r) = r \times \log_x(y) \tag{2.20}$$

Consider a primitive root a for some prime number p (the argument can be developed for nonprimes as well). Then we know that the powers of a from

1 through $(p - 1)$ produce each integer from 1 through $(p - 1)$ exactly once. We also know that any integer b satisfies

$$b \equiv r \pmod{p} \text{ for some } r, \text{ where } 0 \leq r \leq (p - 1)$$

by the definition of modular arithmetic. It follows that for any integer b and a primitive root a of prime number p, we can find a unique exponent i such that

$$b \equiv a^i \pmod{p} \quad \text{where } 0 \leq i \leq (p - 1)$$

This exponent i is referred to as the **discrete logarithm** of the number b for the base $a \pmod{p}$. We denote this value as $\text{dlog}_{a,p}(b)$.[11]

Note the following:

$$\text{dlog}_{a,p}(1) = 0 \text{ because } a^0 \bmod p = 1 \bmod p = 1 \tag{2.21}$$

$$\text{dlog}_{a,p}(a) = 1 \text{ because } a^1 \bmod p = a \tag{2.22}$$

Here is an example using a nonprime modulus, $n = 9$. Here $\phi(n) = 6$ and $a = 2$ is a primitive root. We compute the various powers of a and find

$$\begin{aligned} 2^0 &= 1 & 2^4 &\equiv 7 \pmod 9 \\ 2^1 &= 2 & 2^5 &\equiv 5 \pmod 9 \\ 2^2 &= 4 & 2^6 &\equiv 1 \pmod 9 \\ 2^3 &= 8 \end{aligned}$$

This gives us the following table of the numbers with given discrete logarithms (mod 9) for the root $a = 2$:

Logarithm	0	1	2	3	4	5
Number	1	2	4	8	7	5

To make it easy to obtain the discrete logarithms of a given number, we rearrange the table:

Number	1	2	4	5	7	8
Logarithm	0	1	2	5	4	3

Now consider

$$x = a^{\text{dlog}_{a,p}(x)} \bmod p \qquad y = a^{\text{dlog}_{a,p}(y)} \bmod p$$
$$xy = a^{\text{dlog}_{a,p}(xy)} \bmod p$$

[11] Many texts refer to the discrete logarithm as the **index**. There is no generally agreed notation for this concept, much less an agreed name.

Using the rules of modular multiplication,

$$\begin{aligned} xy \bmod p &= [(x \bmod p)(y \bmod p)] \bmod p \\ a^{\text{dlog}_{a,p}(xy)} \bmod p &= [(a^{\text{dlog}_{a,p}(x)} \bmod p)(a^{\text{dlog}_{a,p}(y)} \bmod p)] \bmod p \\ &= (a^{\text{dlog}_{a,p}(x) + \text{dlog}_{a,p}(y)}) \bmod p \end{aligned}$$

But now consider Euler's theorem, which states that, for every a and n that are relatively prime,

$$a^{\phi(n)} \equiv 1 (\text{mod } n)$$

Any positive integer z can be expressed in the form $z = q + k\phi(n)$, with $0 \leq q < \phi(n)$. Therefore, by Euler's theorem,

$$a^z \equiv a^q (\text{mod } n) \qquad \text{if } z \equiv q \bmod \phi(n)$$

Applying this to the foregoing equality, we have

$$\text{dlog}_{a,p}(xy) \equiv [\text{dlog}_{a,p}(x) + \text{dlog}_{a,p}(y)] (\text{mod } \phi(p))$$

and generalizing,

$$\text{dlog}_{a,p}(y^r) \equiv [r \times \text{dlog}_{a,p}(y)] (\text{mod } \phi(p))$$

This demonstrates the analogy between true logarithms and discrete logarithms.

Keep in mind that unique discrete logarithms mod m to some base a exist only if a is a primitive root of m.

Table 2.8, which is directly derived from Table 2.7, shows the sets of discrete logarithms that can be defined for modulus 19.

Calculation of Discrete Logarithms

Consider the equation

$$y = g^x \bmod p$$

Given g, x, and p, it is a straightforward matter to calculate y. At the worst, we must perform x repeated multiplications, and algorithms exist for achieving greater efficiency (see Chapter 9).

However, given y, g, and p, it is, in general, very difficult to calculate x (take the discrete logarithm). The difficulty seems to be on the same order of magnitude as that of factoring primes required for RSA. At the time of this writing, the asymptotically fastest known algorithm for taking discrete logarithms modulo a prime number is on the order of [BETH91]:

$$e^{((\ln p)^{1/3}(\ln(\ln p))^{2/3})}$$

which is not feasible for large primes.

Table 2.8 Tables of Discrete Logarithms, Modulo 19

(a) Discrete logarithms to the base 2, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{2,19}(a)$	18	1	13	2	16	14	6	3	8	17	12	15	5	7	11	4	10	9

(b) Discrete logarithms to the base 3, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{3,19}(a)$	18	7	1	14	4	8	6	3	2	11	12	15	17	13	5	10	16	9

(c) Discrete logarithms to the base 10, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{10,19}(a)$	18	17	5	16	2	4	12	15	10	1	6	3	13	11	7	14	8	9

(d) Discrete logarithms to the base 13, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{13,19}(a)$	18	11	17	4	14	10	12	15	16	7	6	3	1	5	13	8	2	9

(e) Discrete logarithms to the base 14, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{14,19}(a)$	18	13	7	8	10	2	6	3	14	5	12	15	11	1	17	16	4	9

(f) Discrete logarithms to the base 15, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{15,19}(a)$	18	5	11	10	8	16	12	15	4	13	6	3	7	17	1	2	14	9

2.9 KEY TERMS, REVIEW QUESTIONS, AND PROBLEMS

Key Terms

bijection	Euler's theorem	modulus
composite number	Euler's totient function	order
commutative	Fermat's theorem	prime number
Chinese remainder theorem	greatest common divisor	primitive root
discrete logarithm	identity element	relatively prime
divisor	index	residue
Euclidean algorithm	modular arithmetic	

Review Questions

2.1 What does it mean to say that b is a divisor of a?
2.2 What is the meaning of the expression *a divides b*?
2.3 What is the difference between modular arithmetic and ordinary arithmetic?
2.4 What is a prime number?
2.5 What is Euler's totient function?
2.6 The Miller–Rabin test can determine if a number is not prime but cannot determine if a number is prime. How can such an algorithm be used to test for primality?
2.7 What is a primitive root of a number?
2.8 What is the difference between an index and a discrete logarithm?

Problems

2.1 Reformulate Equation (2.1), removing the restriction that a is a nonnegative integer. That is, let a be any integer.
2.2 Draw a figure similar to Figure 2.1 for $a < 0$.
2.3 For each of the following equations, find an integer x that satisfies the equation.
a. $4x \equiv 2 \pmod 3$
b. $7x \equiv 4 \pmod 9$
c. $5x \equiv 3 \pmod{11}$
2.4 In this text, we assume that the modulus is a positive integer. But the definition of the expression $a \bmod n$ also makes perfect sense if n is negative. Determine the following:
a. $7 \bmod 4$
b. $7 \bmod -4$
c. $-7 \bmod 4$
d. $-7 \bmod -4$
2.5 A modulus of 0 does not fit the definition but is defined by convention as follows: $a \bmod 0 = a$. With this definition in mind, what does the following expression mean: $a \equiv b \pmod 0$?
2.6 In Section 2.3, we define the congruence relationship as follows: Two integers a and b are said to be congruent modulo n if $(a \bmod n) = (b \bmod n)$. We then proved that $a \equiv b \pmod n$ if $n|(a - b)$. Some texts on number theory use this latter relationship as the definition of congruence: Two integers a and b are said to be congruent modulo n if $n|(a - b)$. Using this latter definition as the starting point, prove that, if $(a \bmod n) = (b \bmod n)$, then n divides $(a - b)$.
2.7 What is the smallest positive integer that has exactly k divisors? Provide answers for values for $1 \leq k \leq 8$.
2.8 Prove the following:
a. $a \equiv b \pmod n$ implies $b \equiv a \pmod n$
b. $a \equiv b \pmod n$ and $b \equiv c \pmod n$ imply $a \equiv c \pmod n$
2.9 Prove the following:
a. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$
b. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$
2.10 Find the multiplicative inverse of each nonzero element in Z_5.
2.11 Show that an integer N is congruent modulo 9 to the sum of its decimal digits. For example, $723 \equiv 7 + 2 + 3 \equiv 12 \equiv 1 + 2 \equiv 3 \pmod 9$. This is the basis for the familiar procedure of "casting out 9's" when checking computations in arithmetic.

2.12 **a.** Determine gcd(72345, 43215)
b. Determine gcd(3486, 10292)

2.13 The purpose of this problem is to set an upper bound on the number of iterations of the Euclidean algorithm.
a. Suppose that $m = qn + r$ with $q > 0$ and $0 \leq r < n$. Show that $m/2 > r$.
b. Let A_i be the value of A in the Euclidean algorithm after the ith iteration. Show that

$$A_{i+2} < \frac{A_i}{2}$$

c. Show that if m, n, and N are integers with $(1 \leq m, n, \leq 2^N)$, then the Euclidean algorithm takes at most $2N$ steps to find gcd(m, n).

2.14 The Euclidean algorithm has been known for over 2000 years and has always been a favorite among number theorists. After these many years, there is now a potential competitor, invented by J. Stein in 1961. Stein's algorithms is as follows: Determine gcd(A, B) with $A, B \geq 1$.

STEP 1 Set $A_1 = A, B_1 = B, C_1 = 1$

STEP 2 For $n > 1$, (1) If $A_n = B_n$, stop. gcd$(A, B) = A_nC_n$
(2) If A_n and B_n are both even, set $A_{n+1} = A_n/2$, $B_{n+1} = B_n/2$, $C_{n+1} = 2C_n$
(3) If A_n is even and B_n is odd, set $A_{n+1} = A_n/2$, $B_{n+1} = B_n$, $C_{n+1} = C_n$
(4) If A_n is odd and B_n is even, set $A_{n+1} = A_n$, $B_{n+1} = B_n/2$, $C_{n+1} = C_n$
(5) If A_n and B_n are both odd, set $A_{n+1} = |A_n - B_n|$, $B_{n+1} = \min(B_n, A_n)$, $C_{n+1} = C_n$

Continue to step $n + 1$.
a. To get a feel for the two algorithms, compute gcd(6150, 704) using both the Euclidean and Stein's algorithm.
b. What is the apparent advantage of Stein's algorithm over the Euclidean algorithm?

2.15 **a.** Show that if Stein's algorithm does not stop before the nth step, then

$$C_{n+1} \times \gcd(A_{n+1}, B_{n+1}) = C_n \times \gcd(A_n, B_n)$$

b. Show that if the algorithm does not stop before step $(n - 1)$, then

$$A_{n+2}B_{n+2} \leq \frac{A_nB_n}{2}$$

c. Show that if $1 \leq A, B \leq 2^N$, then Stein's algorithm takes at most $4N$ steps to find gcd(m, n). Thus, Stein's algorithm works in roughly the same number of steps as the Euclidean algorithm.
d. Demonstrate that Stein's algorithm does indeed return gcd(A, B).

2.16 Using the extended Euclidean algorithm, find the multiplicative inverse of
a. 135 mod 61
b. 7465 mod 2464
c. 42828 mod 6407

2.17 The purpose of this problem is to determine how many prime numbers there are. Suppose there are a total of n prime numbers, and we list these in order: $p_1 = 2 < p_2 = 3 < p_3 = 5 < \ldots < p_n$.
a. Define $X = 1 + p_1p_2 \ldots p_n$. That is, X is equal to one plus the product of all the primes. Can we find a prime number P_m that divides X?
b. What can you say about m?
c. Deduce that the total number of primes cannot be finite.
d. Show that $P_{n+1} \leq 1 + p_1p_2 \ldots p_n$.

2.18 The purpose of this problem is to demonstrate that the probability that two random numbers are relatively prime is about 0.6.

a. Let $P = \Pr[\gcd(a, b) = 1]$. Show that $P = \Pr[\gcd(a, b) = d] = P/d^2$. *Hint:* Consider the quantity $\gcd\left(\frac{a}{d}, \frac{b}{d}\right)$.

b. The sum of the result of part (a) over all possible values of d is 1. That is $\Sigma^{d \geq 1} \Pr[\gcd(a, b) = d] = 1$. Use this equality to determine the value of P. *Hint:* Use the identity $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$.

2.19 Why is $\gcd(n, n + 1) = 1$ for two consecutive integers n and $n + 1$?

2.20 Using Fermat's theorem, find $4^{225} \bmod 13$.

2.21 Use Fermat's theorem to find a number a between 0 and 92 with a congruent to 7^{1013} modulo 93.

2.22 Use Fermat's theorem to find a number x between 0 and 37 with x^{73} congruent to 4 modulo 37. (You should not need to use any brute-force searching.)

2.23 Use Euler's theorem to find a number a between 0 and 9 such that a is congruent to 9^{101} modulo 10. (*Note*: This is the same as the last digit of the decimal expansion of 9^{100}.)

2.24 Use Euler's theorem to find a number x between 0 and 14 with x^{61} congruent to 7 modulo 15. (You should not need to use any brute-force searching.)

2.25 Notice in Table 2.6 that $\phi(n)$ is even for $n > 2$. This is true for all $n > 2$. Give a concise argument why this is so.

2.26 Prove the following: If p is prime, then $\phi(p^i) = p^i - p^{i-1}$. *Hint:* What numbers have a factor in common with p^i?

2.27 It can be shown (see any book on number theory) that if $\gcd(m, n) = 1$ then $\phi(mn) = \phi(m)\phi(n)$. Using this property, the property developed in the preceding problem, and the property that $\phi(p) = p - 1$ for p prime, it is straightforward to determine the value of $\phi(n)$ for any n. Determine the following:

a. $\phi(29)$ **b.** $\phi(51)$ **c.** $\phi(455)$ **d.** $\phi(616)$

2.28 It can also be shown that for arbitrary positive integer a, $\phi(a)$ is given by

$$\phi(a) = \prod_{i=1}^{t} [p_i^{a_i - 1}(p_i - 1)]$$

where a is given by Equation (2.9), namely: $a = P_1^{a_1} P_2^{a_2} \ldots P_t^{a_t}$. Demonstrate this result.

2.29 Consider the function: $f(n)$ = number of elements in the set $\{a: 0 \leq a < n$ and $\gcd(a, n) = 1\}$. What is this function?

2.30 Although ancient Chinese mathematicians did good work coming up with their remainder theorem, they did not always get it right. They had a test for primality. The test said that n is prime if and only if n divides $(2^n - 2)$.

a. Give an example that satisfies the condition using an odd prime.

b. The condition is obviously true for $n = 2$. Prove that the condition is true if n is an odd prime (proving the **if** condition).

c. Give an example of an odd n that is not prime and that does not satisfy the condition. You can do this with nonprime numbers up to a very large value. This misled the Chinese mathematicians into thinking that if the condition is true then n is prime.

d. Unfortunately, the ancient Chinese never tried $n = 341$, which is nonprime $(341 = 11 \times 31)$, yet 341 divides $2^{341} - 2$ without remainder. Demonstrate that $2341 \equiv 2 \pmod{341}$ (disproving the **only if** condition). *Hint:* It is not necessary to calculate 2^{341}; play around with the congruences instead.

2.31 Show that, if n is an odd composite integer, then the Miller–Rabin test will return `inconclusive` for $a = 1$ and $a = (n - 1)$.

2.32 If n is composite and passes the Miller–Rabin test for the base a, then n is called a *strong pseudoprime to the base a*. Show that 2047 is a strong pseudoprime to the base 2.

2.33 A common formulation of the Chinese remainder theorem (CRT) is as follows: Let $m_1, \ldots, m_k$ be integers that are pairwise relatively prime for $1 \leq i, j \leq k$, and $i \neq j$. Define M to be the product of all the m_i's. Let $a_1, \ldots, a_k$ be integers. Then the set of congruences:

$$\begin{aligned} x &\equiv a_1 (\text{mod } m_1) \\ x &\equiv a_2 (\text{mod } m_2) \\ &\cdot \\ &\cdot \\ &\cdot \\ x &\equiv a_k (\text{mod } m_k) \end{aligned}$$

has a unique solution modulo M. Show that the theorem stated in this form is true.

2.34 The example used by Sun-Tsu to illustrate the CRT was

$$x \equiv 2 \text{ (mod 3)}; x \equiv 3 \text{ (mod 5)}; x \equiv 2 \text{ (mod 7)}$$

Solve for x.

2.35 Six professors begin courses on Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday, respectively, and announce their intentions of lecturing at intervals of 3, 2, 5, 6, 1, and 4 days, respectively. The regulations of the university forbid Sunday lectures (so that a Sunday lecture must be omitted). When first will all six professors find themselves compelled to omit a lecture? *Hint:* Use the CRT.

2.36 Find all primitive roots of 37.

2.37 Given 5 as a primitive root of 23, construct a table of discrete logarithms, and use it to solve the following congruences.

a. $3x^5 \equiv 2 \text{ (mod 23)}$

b. $7x^{10} + 1 \equiv 0 \text{ (mod 23)}$

c. $5^x \equiv 6 \text{ (mod 23)}$

Programming Problems

2.1 Write a computer program that implements fast exponentiation (successive squaring) modulo n.

2.2 Write a computer program that implements the Miller–Rabin algorithm for a user-specified n. The program should allow the user two choices: (1) specify a possible witness a to test using the Witness procedure or (2) specify a number s of random witnesses for the Miller–Rabin test to check.

APPENDIX 2A THE MEANING OF MOD

The operator mod is used in this book and in the literature in two different ways: as a binary operator and as a congruence relation. This appendix explains the distinction and precisely defines the notation used in this book regarding parentheses. This notation is common but, unfortunately, not universal.

The Binary Operator mod

If a is an integer and n is a positive integer, we define $a \bmod n$ to be the remainder when a is divided by n. The integer n is called the **modulus**, and the remainder is called the **residue**. Thus, for any integer a, we can always write

$$a = \lfloor a/n \rfloor \times n + (a \bmod n)$$

Formally, we define the operator mod as

$$a \bmod n = a - \lfloor a/n \rfloor \times n \quad \text{for } n \neq 0$$

As a binary operation, mod takes two integer arguments and returns the remainder. For example, $7 \bmod 3 = 1$. The arguments may be integers, integer variables, or integer variable expressions. For example, all of the following are valid, with the obvious meanings:

$7 \bmod 3$
$7 \bmod m$
$x \bmod 3$
$x \bmod m$
$(x^2 + y + 1) \bmod (2m + n)$

where all of the variables are integers. In each case, the left-hand term is divided by the right-hand term, and the resulting value is the remainder. Note that if either the left- or right-hand argument is an expression, the expression is parenthesized. The operator mod is not inside parentheses.

In fact, the mod operation also works if the two arguments are arbitrary real numbers, not just integers. In this book, we are concerned only with the integer operation.

The Congruence Relation mod

As a congruence relation, mod expresses that two arguments have the same remainder with respect to a given modulus. For example, $7 \equiv 4 \pmod 3$ expresses the fact that both 7 and 4 have a remainder of 1 when divided by 3. The following two expressions are equivalent:

$$a \equiv b \pmod m \qquad \Leftrightarrow \qquad a \bmod m = b \bmod m$$

Another way of expressing it is to say that the expression $a \equiv b \pmod m$ is the same as saying that $a - b$ is an integral multiple of m. Again, all the arguments may be integers, integer variables, or integer variable expressions. For example, all of the following are valid, with the obvious meanings:

$7 \equiv 4 \pmod 3$
$x \equiv y \pmod m$
$(x^2 + y + 1) \equiv (a + 1) \pmod{[m + n]}$

where all of the variables are integers. Two conventions are used. The congruence sign is $\equiv$. The modulus for the relation is defined by placing the mod operator followed by the modulus in parentheses.

The congruence relation is used to define **residue classes**. Those numbers that have the same remainder r when divided by m form a residue class (mod m). There are m residue classes (mod m). For a given remainder r, the residue class to which it belongs consists of the numbers

$$r, r \pm m, r \pm 2m, \ldots$$

According to our definition, the congruence

$$a \equiv b \pmod{m}$$

signifies that the numbers a and b differ by a multiple of m. Consequently, the congruence can also be expressed in the terms that a and b belong to the same residue class (mod m).