Chapter 8
Polynomials, Curve Fitting, and Interpolation

Polynomials are mathematical expressions that are frequently used for problem solving and modeling in science and engineering. In many cases an equation that is written in the process of solving a problem is a polynomial, and the solution of the problem is the zero of the polynomial. MATLAB has a wide selection of functions that are specifically designed for handling polynomials. How to use polynomials in MATLAB is described in Section 8.1.

Curve fitting is a process of finding a function that can be used to model data. The function does not necessarily pass through any of the points, but models the data with the smallest possible error. There are no limitations to the type of the equations that can be used for curve fitting. Often, however, polynomial, exponential, and power functions are used. In MATLAB curve fitting can be done by writing a program or by interactively analyzing data that is displayed in the Figure Window. Section 8.2 describes how to use MATLAB programming for curve fitting with polynomials and other functions. Section 8.4 describes the basic fitting interface that is used for interactive curve fitting and interpolation.

Interpolation is the process of estimating values between data points. The simplest kind of interpolation is done by drawing a straight line between the points. In a more sophisticated interpolation, data from additional points is used. How to interpolate with MATLAB is discussed in Sections 8.3 and 8.4.

8.1 POLYNOMIALS

Polynomials are functions that have the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

The coefficients $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers, and n which is a nonnegative integer, is the degree, or order, of the polynomial.

Examples of polynomials are:

$f(x) = 5x^5 + 6x^2 + 7x + 3$ polynomial of degree 5.

$f(x) = 2x^2 - 4x + 10$ polynomial of degree 2.

$f(x) = 11x - 5$ polynomial of degree 1.

A constant (e.g., $f(x) = 6$) is a polynomial of degree 0.

In MATLAB, polynomials are represented by a row vector in which the elements are the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$. The first element is the coefficient of the x with the highest power. The vector has to include all the coefficients, including the ones that are equal to 0. For example:

Polynomial	MATLAB representation
$8x + 5$	p = [8 5]
$2x^2 - 4x + 10$	d = [2 –4 10]
$6x^2 - 150$, MATLAB form: $6x^2 + 0x - 150$	h = [6 0 –150]
$5x^5 + 6x^5 - 7x$, MATLAB form: $5x^5 + 0x^4 + 0x^3 + 6x^5 - 7x + 0$	c = [5 0 0 6 7 0]

8.1.1 Value of a Polynomial

The value of a polynomial at a point x can be calculated with the function `polyval` that has the form:

```
polyval(p,x)
```

`p` is a vector with the coefficients of the polynomial.

`x` is a number, or a variable that has an assigned value, or a computable expression.

`x` can also be a vector or a matrix. In such a case the polynomial is calculated for each element (element-by-element), and the answer is a vector, or a matrix, with the corresponding values of the polynomial.

Sample Problem 8-1: Calculating polynomials with MATLAB

For the polynomial $f(x) = x^5 - 12.1x^4 + 40.59x^3 - 17.015x^2 - 71.95x + 35.88$:

(*a*) Calculate $f(9)$.

(*b*) Plot the polynomial for $-1.5 \le x \le 6.7$.

Solution

The problem is solved in the Command Window.

(*a*) The coefficients of the polynomials are assigned to vector `p`. The function `polyval` is then used to calculate the value at $x = 9$.

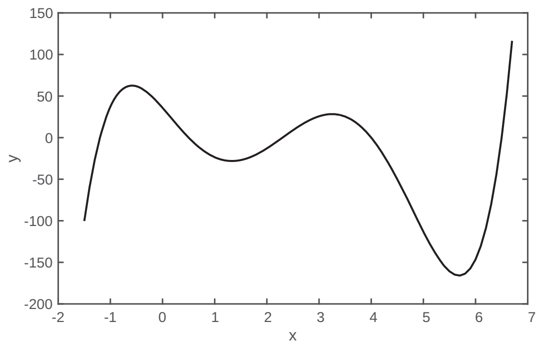
```
>> p = [1 -12.1 40.59 -17.015 -71.95 35.88];
>> polyval(p,9)
ans =
  7.2611e+003
```

(*b*) To plot the polynomial, a vector `x` is first defined with elements ranging from –1.5 to 6.7. Then a vector `y` is created with the values of the polynomial for every element of `x`. Finally, a plot of `y` vs. `x` is made.

```
>> x=-1.5:0.1:6.7;
>> y=polyval(p,x);
>> plot(x,y)
```

Calculating the value of the polynomial for each element of the vector `x`.

The plot created by MATLAB is presented below (axis labels were added with the Plot Editor).



8.1.2 *Roots of a Polynomial*

The roots of a polynomial are the values of the argument for which the value of the polynomial is equal to zero. For example, the roots of the polynomial $f(x) = x^2 - 2x - 3$ are the values of x for which $x^2 - 2x - 3 = 0$, which are $x = -1$ and $x = 3$.

MATLAB has a function, called `roots`, that determines the root, or roots, of a polynomial. The form of the function is:

```
r = roots(p)
```

`r` is a column vector with the roots of the polynomial.

`p` is a row vector with the coefficients of the polynomial.

For example, the roots of the polynomial in Sample Problem 8-1 can be determined by:

```
>> p= 1 -12.1 40.59 -17.015 -71.95 35.88];
>> r=roots(p)
r =
    6.5000
    4.0000
    2.3000
   -1.2000
    0.5000
```

When the roots are known, the polynomial can actually be written as:
$f(x) = (x+1.2)(x-0.5)(x-2.3)(x-4)(x-6.5)$

The `roots` command is very useful for finding the roots of a quadratic equation. For example, to find the roots of $f(x) = 4x^2 + 10x - 8$, type:

```
>> roots([4 10 -8])
ans =
   -3.1375
    0.6375
```

When the roots of a polynomial are known, the `poly` command can be used for determining the coefficients of the polynomial. The form of the `poly` command is:

```
p = poly(r)
```

`p` is a row vector with the coefficients of the polynomial.

`r` is a vector (row or column) with the roots of the polynomial.

For example, the coefficients of the polynomial in Sample Problem 8-1 can be obtained from the roots of the polynomial (see above) by:

```
>> r=[6.5 4 2.3 -1.2 0.5];
>> p=poly(r)
p =
    1.0000  -12.1000   40.5900  -17.0150  -71.9500   35.8800
```

8.1.3 Addition, Multiplication, and Division of Polynomials

Addition:

Two polynomials can be added (or subtracted) by adding (subtracting) the vectors of the coefficients. If the polynomials are not of the same order (which means that the vectors of the coefficients are not of the same length), the shorter vector has to be modified to be of the same length as the longer vector by adding zeros (called padding) in front. For example, the polynomials $f_1(x) = 3x^6 + 15x^5 - 10x^3 - 3x^2 + 15x - 40$ and $f_2(x) = 3x^3 - 2x - 6$ can be added by:

```
>> p1=[3 15 0 -10 -3 15 -40];
>> p2=[3 0 -2 -6];
>> p=p1+[0 0 0 p2]
p =
     3    15     0    -7    -3    13   -46
```

Three 0s are added in front of `p2`, since the order of `p1` is 6 and the order of `p2` is 3.

Multiplication:

Two polynomials can be multiplied using the MATLAB built-in function `conv`, which has the form:

```
c = conv(a,b)
```

`c` is a vector of the coefficients of the polynomial that is the product of the multiplication.

`a` and `b` are the vectors of the coefficients of the polynomials that are being multiplied.

- The two polynomials do not have to be of the same order.
- Multiplication of three or more polynomials is done by using the `conv` function repeatedly.

For example, multiplication of the polynomials $f_1(x)$ and $f_2(x)$ above gives:

```
>> pm=conv(p1,p2)
pm =
     9    45    -6   -78   -99    65   -54   -12   -10   240
```

which means that the answer is:

$$9x^9 + 45x^8 - 6x^7 - 78x^6 - 99x^5 + 65x^4 - 54x^3 - 12x^2 - 10x + 240$$

Division:

A polynomial can be divided by another polynomial with the MATLAB built-in function `deconv`, which has the form:

```
[q,r] = deconv(u,v)
```

`q` is a vector with the coefficients of the quotient polynomial.
`r` is a vector with the coefficients of the remainder polynomial.

`u` is a vector with the coefficients of the numerator polynomial.
`v` is a vector with the coefficients of the denominator polynomial.

For example, dividing $2x^3 + 9x^2 + 7x - 6$ by $x + 3$ is done by:

```
>> u=[2 9 7 -6];
>> v=[1 3];
```

```
>> [a b]=deconv(u,v)
a =
     2     3    -2
b =
     0     0     0     0
```

The answer is: $2x^2 + 3x - 2$.

Remainder is zero.

An example of division that gives a remainder is $2x^6 - 13x^5 + 75x^3 + 2x^2 - 60$ divided by $x^2 - 5$:

```
>> w=[2 -13 0 75 2 0 -60];
>> z=[1 0 -5];
>> [g h]=deconv(w,z)
g =
    2   -13    10    10    52
h =
    0     0     0     0     0    50   200
```

The quotient is: $2x^4 - 13x^3 + 10x^2 + 10x + 52$.

The remainder is: $50x + 200$.

The answer is: $2x^4 - 13x^3 + 10x^2 + 10x + 52 + \frac{50x+200}{x^2-5}$.

8.1.4 Derivatives of Polynomials

The built-in function `polyder` can be used to calculate the derivative of a single polynomial, a product of two polynomials, or a quotient of two polynomials, as shown in the following three commands.

`k = polyder(p)` Derivative of a single polynomial. `p` is a vector with the coefficients of the polynomial. `k` is a vector with the coefficients of the polynomial that is the derivative.

`k = polyder(a,b)` Derivative of a product of two polynomials. `a` and `b` are vectors with the coefficients of the polynomials that are multiplied. `k` is a vector with the coefficients of the polynomial that is the derivative of the product.

`[n d] = polyder(u,v)` Derivative of a quotient of two polynomials. `u` and `v` are vectors with the coefficients of the numerator and denominator polynomials. `n` and `d` are vectors with the coefficients of the numerator and denominator polynomials in the quotient that is the derivative.

The only difference between the last two commands is the number of output arguments. With two output arguments MATLAB calculates the derivative of the quotient of two polynomials. With one output argument, the derivative is of the product.

For example, if $f_1(x) = 3x^2 - 2x + 4$, and $f_2(x) = x^2 + 5$, the derivatives of $3x^2 - 2x + 4$, $(3x^2 - 2x + 4)(x^2 + 5)$, and $\frac{3x^2-2x+4}{x^2+5}$ can be determined by:

```
>> f1= 3 -2 4];
>> f2=[1 0 5];
>> k=polyder(f1)
k =
     6    -2
>> d=polyder(f1,f2)
d =
    12    -6    38   -10
>> [n d]=polyder(f1,f2)
n =
     2    22   -10
d =
     1     0    10     0    25
```

Creating the vectors of coefficients of f_1 and f_2.

The derivative of f_1 is: $6x - 2$.

The derivative of f_1*f_2 is: $12x^3 - 6x^2 + 38x - 10$.

The derivative of $\frac{3x^2-2x+4}{x^2+5}$ is: $\frac{2x^2+22x-10}{x^4+10x^2+25}$.

8.2 *CURVE FITTING*

Curve fitting, also called regression analysis, is a process of fitting a function to a set of data points. The function can then be used as a mathematical model of the data. Since there are many types of functions (linear, polynomial, power, exponential, etc.), curve fitting can be a complicated process. Many times one has some idea of the type of function that might fit the given data and will need only to determine the coefficients of the function. In other situations, where nothing is known about the data, it is possible to make different types of plots that provide information about possible forms of functions that might fit the data well. This section describes some of the basic techniques for curve fitting and the tools that MATLAB has for this purpose.

8.2.1 Curve Fitting with Polynomials; The `polyfit` *Function*

Polynomials can be used to fit data points in two ways. In one the polynomial passes through all the data points, and in the other the polynomial does not necessarily pass through any of the points but overall gives a good approximation of the data. The two options are described below.

Polynomials that pass through all the points:

When n points (x_i, y_i) are given, it is possible to write a polynomial of degree $n-1$ that passes through all the points. For example, if two points are given it is possible to write a linear equation in the form of $y = mx + b$ that passes through the points. With three points, the equation has the form of

$y = ax^2 + bx + c$. With n points the polynomial has the form $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$. The coefficients of the polynomial are determined by substituting each point in the polynomial and then solving the n equations for the coefficients. As will be shown later in this section, polynomials of high degree might give a large error if they are used to estimate values between data points.

Polynomials that do not necessarily pass through any of the points:

When n points are given, it is possible to write a polynomial of degree less than $n-1$ that does not necessarily pass through any of the points but that overall approximates the data. The most common method of finding the best fit to data points is the method of least squares. In this method, the coefficients of the polynomial are determined by minimizing the sum of the squares of the residuals at all the data points. The residual at each point is defined as the difference between the value of the polynomial and the value of the data. For example, consider the case of finding the equation of a straight line that best fits four data points as shown in Figure 8-1. The points are (x_1,y_1), (x_2,y_2), (x_3,y_3), and

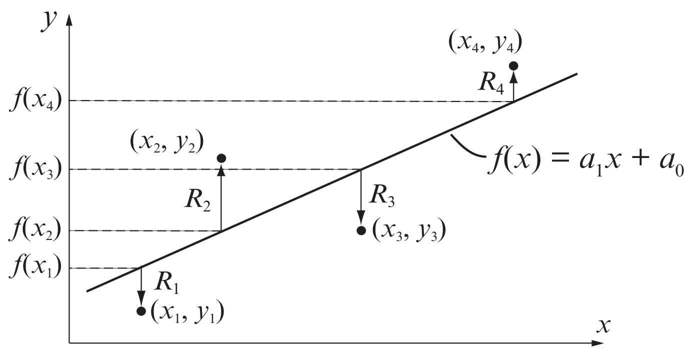


Figure 8-1: Least squares fitting of first-degree polynomial to four points.

(x_4,y_4), and the polynomial of the first degree can be written as $f(x) = a_1x + a_0$. The residual, R_i, at each point is the difference between the value of the function at x_i and y_i, $R_i = f(x_i) - y_i$. An equation for the sum of the squares of the residuals R_i of all the points is given by:

$$R = [f(x_1) - y_1]^2 + [f(x_2) - y_2]^2 + [f(x_3) - y_3]^2 + [f(x_4) - y_4]^2$$

or, after substituting the equation of the polynomial at each point, by:

$$R = [a_1x_1 + a_0 - y_1]^2 + [a_1x_2 + a_0 - y_2]^2 + [a_1x_3 + a_0 - y_3]^2 + [a_1x_4 + a_0 - y_4]^2$$

At this stage R is a function of a_1 and a_0. The minimum of R can be determined by taking the partial derivative of R with respect to a_1 and a_0 (two equations) and equating them to zero:

$$\frac{\partial R}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial R}{\partial a_0} = 0$$

This results in a system of two equations with two unknowns, a_1 and a_0. The solution of these equations gives the values of the coefficients of the polynomial that best fits the data. The same procedure can be followed with more points and higher-order polynomials. More details on the least squares method can be found in books on numerical analysis.

Curve fitting with polynomials is done in MATLAB with the `polyfit` function, which uses the least squares method. The basic form of the `polyfit` function is:

```
p = polyfit(x,y,n)
```

`p` is the vector of the coefficients of the polynomial that fits the data.

`x` is a vector with the horizontal coordinates of the data points (independent variable).
`y` is a vector with the vertical coordinates of the data points (dependent variable).
`n` is the degree of the polynomial.

For the same set of m points, the `polyfit` function can be used to fit polynomials of any order up to $m-1$. If n = 1 the polynomial is a straight line, if n = 2 the polynomial is a parabola, and so on. The polynomial passes through all the points if $n = m - 1$ (the order of the polynomial is one less than the number of points). It should be pointed out here that a polynomial that passes through all the points, or polynomials with higher order, do not necessarily give a better fit overall. High-order polynomials can deviate significantly between the data points.

Figure 8-2 shows how polynomials of different degrees fit the same set of data points. A set of seven points is given by (0.9, 0.9), (1.5, 1.5), (3, 2.5), (4, 5.1),

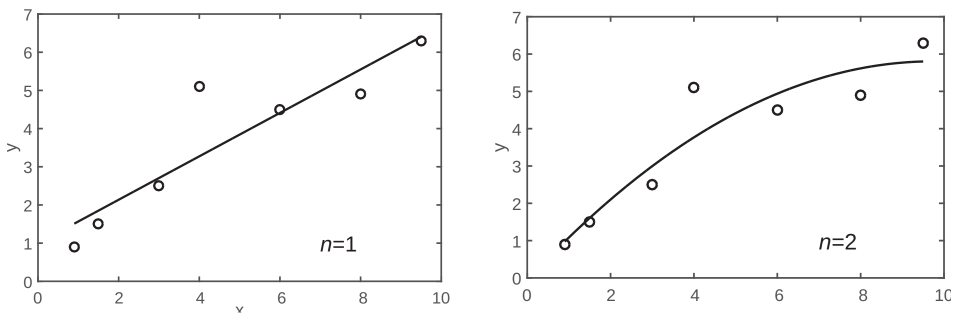


Figure 8-2: Fitting data with polynomials of different order.

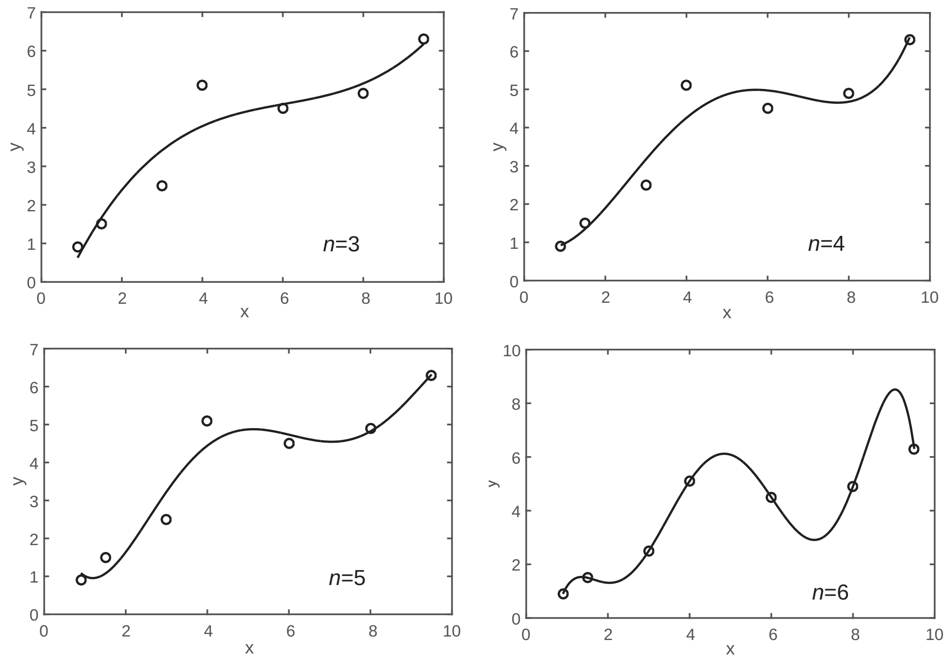


Figure 8-2: Fitting data with polynomials of different order. (Continued)

(6, 4.5), (8, 4.9), and (9.5, 6.3). The points are fitted using the `polyfit` function with polynomials of degrees 1 through 6. Each plot in Figure 8-2 shows the same data points, marked with circles, and a curve-fitted line that corresponds to a polynomial of the specified degree. It can be seen that the polynomial with $n = 1$ is a straight line, and that with $n = 2$ is a slightly curved line. As the degree of the polynomial increases, the line develops more bends such that it passes closer to more points. When $n = 6$, which is one less than the number of points, the line passes through all the points. However, between some of the points, the line deviates significantly from the trend of the data.

The script file used to generate one of the plots in Figure 8-2 (the polynomial with $n = 3$) is shown below. Note that in order to plot the polynomial (the line), a new vector `xp` with small spacing is created. This vector is then used

`x=[0.9 1.5 3 4 6 8 9.5];` `y=[0.9 1.5 2.5 5.1 4.5 4.9 6.3];`	Create vectors `x` and `y` with the coordinates of the data points.
`p=polyfit(x,y,3)`	Create a vector `p` using the `polyfit` function.
`xp=0.9:0.1:9.5;`	Create a vector `xp` to be used for plotting the polynomial.
`yp=polyval(p,xp)`	Create a vector `yp` with values of the polynomial at each `xp`.
`plot(x,y,'o',xp,yp)`	A plot of the seven points and the polynomial.
`xlabel('x'); ylabel('y')`	

with the function `polyval` to create a vector `yp` with the value of the polynomial for each element of `xp`.

When the script file is executed, the following vector `p` is displayed in the Command Window.

```
p =
    0.0220   -0.4005    2.6138   -1.4158
```

This means that the polynomial of the third degree in Figure 8-2 has the form $0.022x^3 - 0.4005x^2 + 2.6138x - 1.4148$.

8.2.2 Curve Fitting with Functions Other than Polynomials

Many situations in science and engineering require fitting functions that are not polynomials to given data. Theoretically, any function can be used to model data within some range. For a particular data set, however, some functions provide a better fit than others. In addition, determining the best-fitting coefficients can be more difficult for some functions than for others. This section covers curve fitting with power, exponential, logarithmic, and reciprocal functions, which are commonly used. The forms of these functions are:

$y = bx^m$ (power function)

$y = be^{mx}$ or $y = b10^{mx}$ (exponential function)

$y = m\ln(x) + b$ or $y = m\log(x) + b$ (logarithmic function)

$y = \frac{1}{mx+b}$ (reciprocal function)

All of these functions can easily be fitted to given data with the `polyfit` function. This is done by rewriting the functions in a form that can be fitted with a linear polynomial ($n = 1$), which is

$$y = mx + b$$

The logarithmic function is already in this form, and the power, exponential, and reciprocal equations can be rewritten as:

$\ln(y) = m\ln(x) + \ln b$ (power function)

$\ln(y) = mx + \ln(b)$ or $\log(y) = mx + \log(b)$ (exponential function)

$\frac{1}{y} = mx + b$ (reciprocal function)

These equations describe a linear relationship between $\ln(y)$ and $\ln(x)$ for the power function, between $\ln(y)$ and x for the exponential function, between y and $\ln(x)$ or $\log(x)$ for the logarithmic function, and between $1/y$ and x for the reciprocal function. This means that the `polyfit(x,y,1)` function can be used to determine the best-fit constants m and b for best fit if, instead of `x` and `y`, the following arguments are used.

Function		`polyfit` **function form**
power	$y = bx^m$	`p=polyfit(log(x),log(y),1)`
exponential	$y = be^{mx}$ or $y = b10^{mx}$	`p=polyfit(x,log(y),1)` or `p=polyfit(x,log10(y),1)`
logarithmic	$y = m\ln(x) + b$ or $y = m\log(x) + b$	`p=polyfit(log(x),y,1)` or `p=polyfit(log10(x),y,1)`
reciprocal	$y = \frac{1}{mx+b}$	`p=polyfit(x,1./y,1)`

The result of the `polyfit` function is assigned to `p`, which is a two-element vector. The first element, `p(1)`, is the constant `m`, and the second element, `p(2)`, is `b` for the logarithmic and reciprocal functions, $\ln(b)$ or $\log(b)$ for the exponential function, and $\ln(b)$ for the power function ($b = e^{p(2)}$ or $b = 10^{p(2)}$ for the exponential function, and $b = e^{p(2)}$ for the power function).

For given data it is possible to estimate, to some extent, which of the functions has the potential for providing a good fit. This is done by plotting the data using different combinations of linear and logarithmic axes. If the data points in one of the plots appear to fit a straight line, the corresponding function can provide a good fit according to the list below.

***x* axis**	***y* axis**	**Function**
linear	linear	linear $y = mx + b$
logarithmic	logarithmic	power $y = bx^m$
linear	logarithmic	exponential $y = be^{mx}$ or $y = b10^{mx}$
logarithmic	linear	logarithmic $y = m\ln(x) + b$ or $y = m\log(x) + b$
linear	linear (plot $1/y$)	reciprocal $y = \frac{1}{mx+b}$

Other considerations in choosing a function:

- Exponential functions cannot pass through the origin.
- Exponential functions can fit only data with all positive y's or all negative y's.
- Logarithmic functions cannot model $x = 0$ or negative values of x.
- For the power function $y = 0$ when $x = 0$.
- The reciprocal equation cannot model $y = 0$.

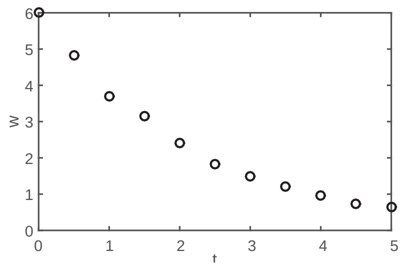
The following example illustrates the process of fitting a function to a set of data points.

Sample Problem 8-2: Fitting an equation to data points

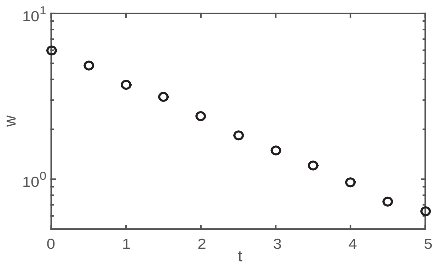
The following data points are given. Determine a function $w = f(t)$ (t is the independent variable, w is the dependent variable) with a form discussed in this section that best fits the data.

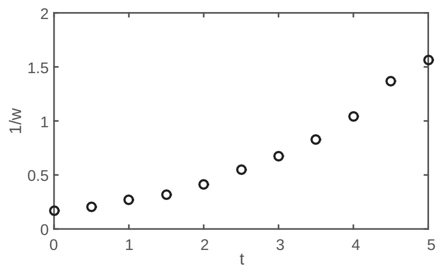
t	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
w	6.00	4.83	3.70	3.15	2.41	1.83	1.49	1.21	0.96	0.73	0.64

Solution



The data is first plotted with linear scales on both axes. The figure indicates that a linear function will not give the best fit since the points do not appear to line up along a straight line. From the other possible functions, the logarithmic function is excluded since for the first point $t = 0$, and the power function is excluded since at $t = 0$, $w \neq 0$. To check if the other two functions (exponential and reciprocal) might give a better fit, two additional plots, shown below, are made. The plot on the left has a log scale on the vertical axis and linear horizontal axis. In the plot on the right, both axes have linear scales, and the quantity $1/w$ is plotted on the vertical axis.





In the left figure, the data points appear to line up along a straight line. This indicates that an exponential function of the form $y = be^{mx}$ can give a good fit to the data. A program in a script file that determines the constants b and m, and that plots the data points and the function is given below.

```
t=0:0.5:5;
w=[6 4.83 3.7 3.15 2.41 1.83 1.49 1.21 0.96 0.73 0.64];
p=polyfit(t,log(w),1);
```

Create vectors `t` and `w` with the coordinates of the data points.

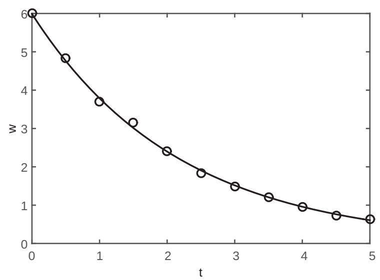
Use the `polyfit` function with `t` and `log(w)`.

```
m=p(1)
b=exp(p(2))                    Determine the coefficient b.
tm=0:0.1:5;      Create a vector tm to be used for plotting the polynomial.
wm=b*exp(m*tm);       Calculate the function value at each element of tm.
plot(t,w,'o',tm,wm)            Plot the data points and the function.
```

When the program is executed, the values of the constants m and b are displayed in the Command Window.

```
m =
   -0.4580
b =
    5.9889
```

The plot generated by the program, which shows the data points and the function (with axis labels added with the Plot Editor) is



It should be pointed out here that in addition to the power, exponential, logarithmic, and reciprocal functions that are discussed in this section, many other functions can be written in a form suitable for curve fitting with the `polyfit` function. One example where a function of the form $y = e^{(a_2x^2 + a_1x + a_0)}$ is fitted to data points using the `polyfit` function with a third-order polynomial is described in Sample Problem 8-7.

8.3 *INTERPOLATION*

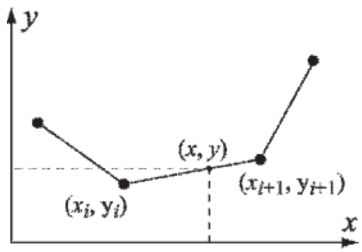
Interpolation is the estimation of values between data points. MATLAB has interpolation functions that are based on polynomials, which are described in this section, and on Fourier transformation, which is outside the scope of this book. In one-dimensional interpolation, each point has one independent variable (x) and one dependent variable (y). In two-dimensional interpolation, each point has two independent variables (x and y) and one dependent variable (z).

One-dimensional interpolation:

If only two data points exist, the points can be connected with a straight line and a linear equation (polynomial of first order) can be used to estimate values between the points. As was discussed in the previous section, if three (or four) data points exist, a second- (or a third-) order polynomial that passes through the points can be determined and then be used to estimate values between the points. As the number of points increases, a higher-order polynomial is required for the polynomial to pass through all the points. Such a polynomial, however, will not necessarily give a good approximation of the values between the points. This is illustrated in Figure 8-2 with $n = 6$.

A more accurate interpolation can be obtained if instead of considering all the points in the data set (by using one polynomial that passes through all the points), only a few data points in the neighborhood where the interpolation is needed are considered. In this method, called spline interpolation, many low-order polynomials are used, where each is valid only in a small domain of the data set.

The simplest method of spline interpolation is called linear spline interpolation. In this method, shown on the right, every two adjacent points are connected with a straight line (a polynomial of first degree). The equation of a straight line that passes through two adjacent points (x_i, y_j) and (x_{i+1}, y_{j+1}) and that can be used to calculate the value of y for any x between the points is given by:



$$y = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}x + \frac{y_i x_{i+1} - y_{i+1} x_i}{x_{i+1} - x_i}$$

In a linear interpolation, the line between two data points has a constant slope, and there is a change in the slope at every point. A smoother interpolation curve can be obtained by using quadratic or cubic polynomials. In these methods, called quadratic splines and cubic splines, a second-, or third-order polynomial is used to interpolate between every two points. The coefficients of the polynomial are determined by using data from points that are adjacent to the two data points. The theoretical background for the determination of the constants of the polynomials is beyond the scope of this book and can be found in books on numerical analysis.

One-dimensional interpolation in MATLAB is done with the `interp1` (the last character is the numeral one) function, which has the form:

```
yi = interp1(x,y,xi,'method')
```

`yi` is the interpolated value.

`x` is a vector with the horizontal coordinates of the input data points (independent variable).
`y` is a vector with the vertical coordinates of the input data points (dependent variable).
`xi` is the horizontal coordinate of the interpolation point (independent variable).

Method of interpolation, typed as a string (optional).

- The vector `x` must be monotonic (with elements in ascending or descending order).
- `xi` can be a scalar (interpolation of one point) or a vector (interpolation of many points). `yi` is a scalar or a vector with the corresponding interpolated values.
- MATLAB can do the interpolation using one of several methods that can be specified. These methods include:

`'nearest'`	returns the value of the data point that is nearest to the interpolated point.
`'linear'`	uses linear spline interpolation.
`'spline'`	uses cubic spline interpolation.
`'pchip'`	uses piecewise cubic Hermite interpolation, also called `'cubic'`

- When the `'nearest'` and the `'linear'` methods are used, the value(s) of `xi` must be within the domain of `x`. If the `'spline'` or the `'pchip'` methods are used, `xi` can have values outside the domain of `x` and the function `interp1` performs extrapolation.
- The `'spline'` method can give large errors if the input data points are nonuniform such that some points are much closer together than others.
- Specification of the method is optional. If no method is specified, the default is `'linear'`.

Sample Problem 8-3: Interpolation

The following data points, which are points of the function $f(x) = 1.5^x \cos(2x)$, are given. Use linear, spline, and pchip interpolation methods to calculate the value of y between the points. Make a figure for each of the interpolation methods. In the figure show the points, a plot of the function, and a curve that corresponds to the interpolation method.

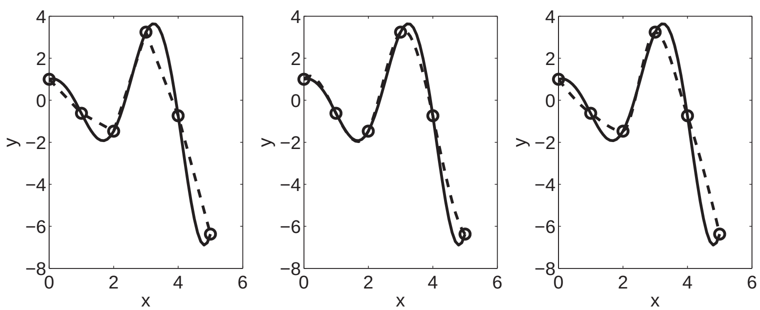
x	0	1	2	3	4	5
y	1.0	–0.6242	–1.4707	3.2406	–0.7366	–6.3717

Solution

The following is a program written in a script file that solves the problem:

```
x=0:1.0:5;            Create vectors x and y with coordinates of the data points.
y=[1.0 -0.6242 -1.4707 3.2406 -0.7366 -6.3717];
xi=0:0.1:5;           Create vector xi with points for interpolation.
yilin=interp1(x,y,xi,'linear');   Calculate y points from linear interpolation.
yispl=interp1(x,y,xi,'spline');   Calculate y points from spline interpolation.
yipch=interp1(x,y,xi,'pchip');    Calculate y points from pchip interpolation.
yfun=1.5.^xi.*cos(2*xi);          Calculate y points from the function.
subplot(1,3,1)
plot(x,y,'o',xi,yfun,xi,yilin,'--');
subplot(1,3,2)
plot(x,y,'o',xi,yfun,xi,yispl,'--');
subplot(1,3,3)
plot(x,y,'o',xi,yfun,xi,yipch,'--');
```

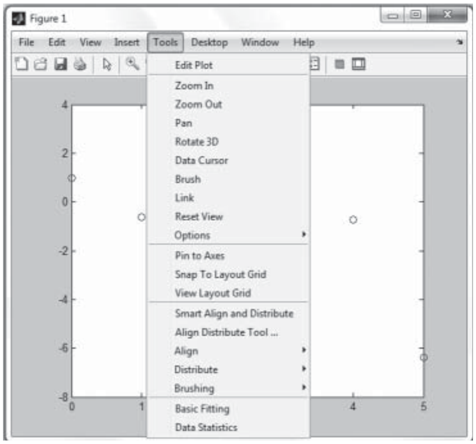
The three figures generated by the program are shown below (axes labels were added with the Plot Editor). The data points are marked with circles, the interpolation curves are plotted with dashed lines, and the function is shown with a solid line. The left figure shows the linear interpolation, the middle is the spline, and the figure on the right shows the pchip interpolation.



8.4 *The Basic Fitting Interface*

The basic fitting interface is a tool that can be used to perform curve fitting and interpolation interactively. By using the interface the user can:

- Curve-fit the data points with polynomials of various degrees up to 10, and with spline and Hermite interpolation methods.
- Plot the various fits on the same graph so that they can be compared.
- Plot the residuals of the various polynomial fits and compare the norms of the residuals.
- Calculate the values of specific points with the various fits.
- Add the equations of the polynomials to the plot.



To activate the basic fitting interface, the user first has to make a plot of the data points. Then the interface is activated by selecting **Basic Fitting** in the **Tools** menu, as shown on the right. This opens the Basic Fitting Window, shown in Figure 8-3. When the window first opens, only one panel (the **Plot fits** panel) is visible. The window can be extended to show a second panel (the **Numerical results** panel) by clicking on the → button. One click adds the first section of the panel, and a second click makes the window look as shown in Figure 8-3. The window can be reduced back by clicking on the ← button. The first two items in the Basic Fitting Window are related to the selection of the data points:

Select data: Used to select a specific set of data points for curve fitting in a figure that has more than one set of data points. Only one set of data points can be curve-fitted at a time, but multiple fits can be performed simultaneously on the same set.

Center and scale x data: When this box is checked, the data is centered at zero mean and scaled to unit standard deviation. This might be needed in order to improve the accuracy of numerical computation.

The next four items are in the **Plot fits** panel and are related to the display of the fit.

Check to display fits on figure: The user selects the fits to be displayed in the figure. The selections include interpolation with spline interpolant (interpolation method) that uses the `spline` function, interpolation with Hermite interpolant that uses the `pchip` function, and polynomials of various degrees

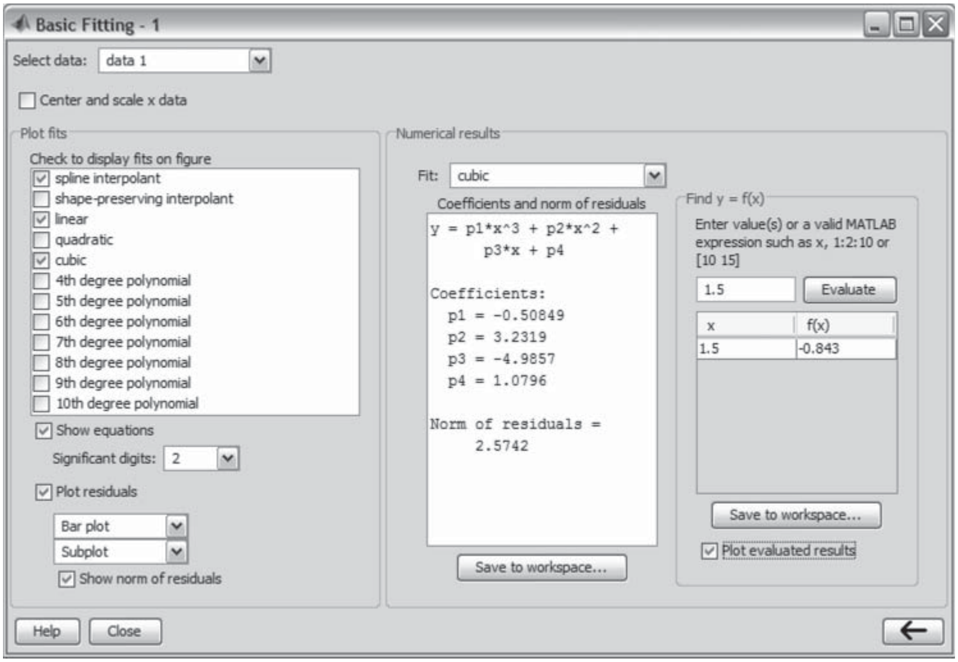


Figure 8-3: The Basic Fitting Window.

that use the `polyfit` function. Several fits can be selected and displayed simultaneously.

Show equations: When this box is checked, the equations of the polynomials that were selected for the fit are displayed in the figure. The equations are displayed with the number of significant digits selected in the adjacent sign menu.

Plot residuals: When this box is checked, a plot that shows the residual at each data point is created (residuals are defined in Section 8.2.1). Choices in the menus include a bar plot, a scatter plot, and a line plot that can be displayed as a subplot in the same Figure Window that has the plot of the data points or as a separate plot in a different Figure Window.

Show norm of residuals: When this box is checked, the norm of the residuals is displayed in the plot of the residuals. The norm of the residual is a measure of the quality of the fit. A smaller norm corresponds to a better fit.

The next three items are in the **Numerical results** panel. They provide the numerical information for one fit, independently of the fits that are displayed:

Fit: The user selects the fit to be examined numerically. The fit is shown on the plot only if it is selected in the **Plot fit** panel.

Coefficients and norm of residuals: Displays the numerical results for the polynomial fit that is selected in the **Fit** menu. It includes the coefficients of the polynomial and the norm of the residuals. The results can be saved by clicking on the **Save to workspace** button.

Find y = f(x): Provides a means for obtaining interpolated (or extrapolated) numerical values for specified values of the independent variable. Enter the value of the independent variable in the box, and click on the **Evaluate** button. When the **Plot evaluated results** box is checked, the point is displayed on the plot.

As an example, the basic fitting interface is used for fitting the data points from Sample Problem 8-3. The Basic Fitting Window is the one shown in Figure

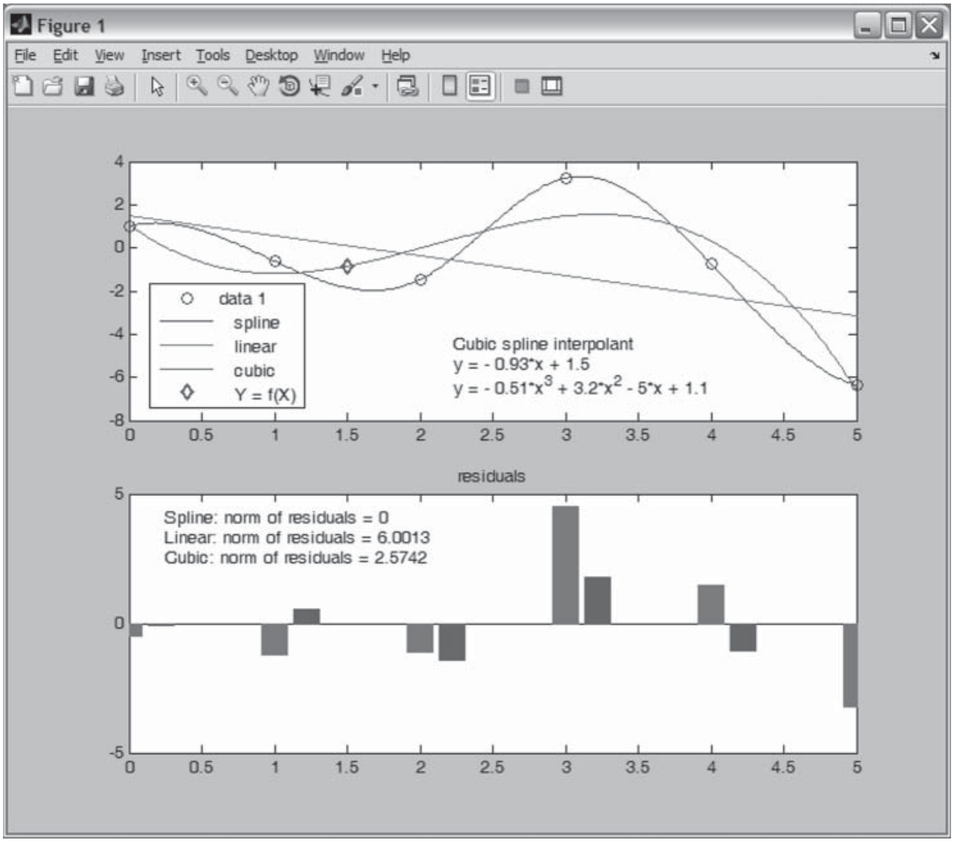


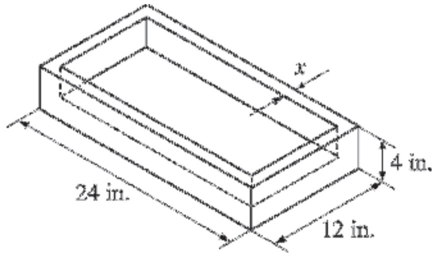
Figure 8-4: A Figure Window modified by the Basic Fitting Interface.

8-3, and the corresponding Figure Window is shown in Figure 8-4. The Figure Window includes a plot of the points, one interpolation fit (spline), two polynomial fits (linear and cubic), a display of the equations of the polynomial fits, and a mark of the point $x = 1.5$ that is entered in the **Find y = f(x)** box of the Basic Fitting Window. The Figure Window also includes a plot of the residuals of the polynomial fits and a display of their norm.

8.5 Examples of MATLAB Applications

Sample Problem 8-4: Determining wall thickness of a box

The outside dimensions of a rectangular box (bottom and four sides, no top), made of aluminum, are 24 by 12 by 4 inches. The wall thickness of the bottom and the sides is x. Derive an expression that relates the weight of the box and the wall thickness x. Determine the thickness x for a box that weighs 15 lb. The specific weight of aluminum is 0.101 lb/in.^{3}.



Solution

The volume of the aluminum V_{Al} is calculated from the weight W of the box by:

$$V_{Al} = \frac{W}{\gamma}$$

where γ is the specific weight. The volume of the aluminum based on the dimensions of the box is given by

$$V_{Al} = 24 \cdot 12 \cdot 4 - (24 - 2x)(12 - 2x)(4 - x)$$

where the inside volume of the box is subtracted from the outside volume. This equation can be rewritten as

$$(24 - 2x)(12 - 2x)(4 - x) + V_{Al} - 24 \cdot 12 \cdot 4 = 0$$

which is a third-degree polynomial. A root of this polynomial is the required thickness x. A program in a script file that determines the polynomial and solves for the roots is:

`W=15; gamma=0.101;`	Assign `W` and `gamma`.
`VAlum=W/gamma;`	Calculate the volume of the aluminum.
`a=[-2  24];`	Assign the polynomial $24 - 2x$ to `a`.
`b=[-2  12];`	Assign the polynomial $12 - 2x$ to `b`.
`c=[-1  4];`	Assign the polynomial $4 - x$ to `c`.
`Vin=conv(c, conv(a,b));`	Multiply the three polynomials above.
`polyeq=[0 0 0 (VAlum-24*12*4)]+Vin`	Add V_{Al}– 24*12*4 to `Vin`.
`x=roots(polyeq)`	Determine the roots of the polynomial.

Note in the second-to-last line that in order to add the quantity $V_{Al} - 24 \cdot 12 \cdot 4$ to the polynomial `Vin` it has to be written as a polynomial of the same order as `Vin` (`Vin` is a polynomial of third order). When the program (saved as Chap8SamPro4) is executed, the coefficients of the polynomial and the value of x are displayed:

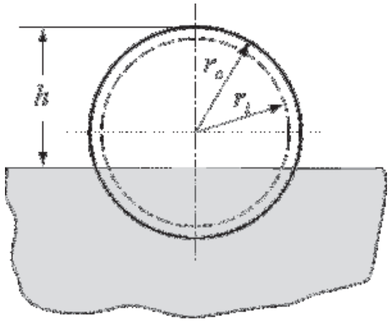
```
>> Chap8SamPro4
polyeq =
 -4.0000   88.0000 -576.0000   148.514
x =
  10.8656 + 4.4831i
  10.8656 - 4.4831i
  0.2687
```

The polynomial is: $-4x^3 + 88x^2 - 576x + 148.515$.

The polynomial has one real root, $x = 0.2687$ in., which is the thickness of the aluminum

Sample Problem 8-5: Floating height of a buoy

An aluminum thin-walled sphere is used as a marker buoy. The sphere has a radius of 60 cm and a wall thickness of 12 mm. The density of aluminum is $\rho_{Al} = 2690$ kg/m^3. The buoy is placed in the ocean, where the density of the water is 1030 kg/m^3. Determine the height h between the top of the buoy and the surface of the water.



Solution

According to Archimedes's law, the buoyancy force applied to an object that is placed in a fluid is equal to the weight of the fluid that is displaced by the object. Accordingly, the aluminum sphere will be at a depth such that the weight of the sphere is equal to the weight of the fluid displaced by the part of the sphere that is submerged.

The weight of the sphere is given by

$$W_{sph} = \rho_{Al} V_{Al}\, g = \rho_{Al} \frac{4}{3}\pi(r_o^3 - r_i^3)g$$

where V_{Al} is the volume of the aluminum; r_o and r_i are the outside and inside radii of the sphere, respectively; and g is the gravitational acceleration.

The weight of the water that is displaced by the spherical portion that is submerged is given by:

$$W_{wtr} = \rho_{wtr} V_{wtr}\, g = \rho_{wtr} \frac{1}{3}\pi(2r_o - h)^2(r_o + h)g$$

Setting the two weights equal to each other gives the following equation:

$$h^3 - 3r_o h^2 + 4r_o^3 - 4\frac{\rho_{Al}}{\rho_{wtr}}(r_o^3 - r_i^3) = 0$$

The last equation is a third-degree polynomial for h. The root of the polynomial is the answer.

A solution with MATLAB is obtained by writing the polynomials and using the `roots` function to determine the value of h.This is done in the following script file:

```
rout=0.60; rin=0.588;                         Assign the radii to variables.
rhoalum=2690; rhowtr=1030;                    Assign the densities to variables.
a0=4*rout^3-4*rhoalum*(rout^3-rin^3)/rhowtr;  Assign the coefficient a0.
p = [1 -3*rout 0 a0];                         Assign the coefficient vector of the polynomial.
h = roots(p)                                  Calculate the roots of the polynomial.
```

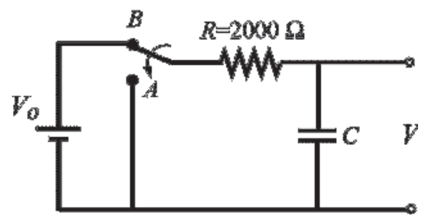
When the script file is executed in the Command Window, as shown below, the answer is three roots, since the polynomial is of the third degree. The only answer that is physically possible is the second, where h = 0.9029 m.

```
>> Chap8SamPro5
h =
    1.4542
    0.9029
   -0.5570
```

The polynomial has three roots. The only one that is physically possible for the problem is 0.9029 m.

Sample Problem 8-6: Determining the size of a capacitor

An electrical capacitor has an unknown capacitance. In order to determine its capacitance, the capacitor is connected to the circuit shown. The switch is first connected to B and the capacitor is charged. Then, the switch is connected to A and the capacitor discharges through the resistor.



As the capacitor is discharging, the voltage across the capacitor is measured for 10 s in intervals of 1 s. The recorded measurements are given in the table below. Plot the voltage as a function of time and determine the capacitance of the capacitor by fitting an exponential curve to the data points.

t (s)	1	2	3	4	5	6	7	8	9	10
V (V)	9.4	7.31	5.15	3.55	2.81	2.04	1.26	0.97	0.74	0.58

Solution

When a capacitor discharges through a resistor, the voltage of the capacitor as a function of time is given by

$$V = V_0 e^{-t/RC}$$

where V_0 is the initial voltage, R the resistance of the resistor, and C the capacitance of the capacitor. As was explained in Section 8.2.2 the exponential function can be written as a linear equation for $\ln(V)$ and t in the form:

$$\ln(V) = \frac{-1}{RC}t + \ln(V_0)$$

This equation, which has the form $y = mx + b$, can be fitted to the data points by using the `polyfit(x,y,1)` function with t as the independent variable `x` and $\ln(V)$ as the dependent variable `y`. The coefficients m and b determined by the `polyfit` function are then used to determine C and V_0 by:

$$C = \frac{-t}{Rm} \quad \text{and} \quad V_0 = e^b$$

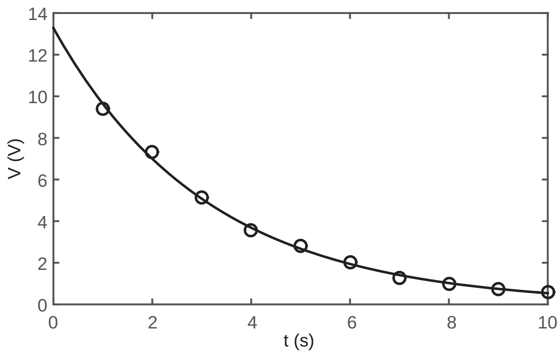
The following program written in a script file determines the best-fit exponential function to the data points, determines C and V_0, and plots the points and the fitted function.

```
R=2000;                                    Define R.
t=1:10;                                    Assign the data points to vectors t and v.
v=[9.4 7.31 5.15 3.55 2.81 2.04 1.26 0.97 0.74 0.58];
p=polyfit(t,log(v),1);                     Use the polyfit function with t and log(v).
C=-1/(R*p(1))                              Calculate C from p(1), which is m in the equation.
V0=exp(p(2))                               Calculate V0 from p(2), which is b in the equation.
tplot=0:0.1:10;                            Create vector tplot of time for plotting the function.
vplot=V0*exp(-tplot./(R*C));               Create vector vplot for plotting the function.
plot(t,v,'o',tplot,vplot)
```

When the script file is executed (saved as Chap8SamPro6) the values of C and V_0 are displayed in the Command Window as shown below:

```
>> Chap8SamPro6
C =                                        The capacitance of the capacitor is 1,600 μF.
    0.0016
V0 =
   13.2796
```

The program creates also the following plot (axis labels were added to the plot using the Plot Editor):



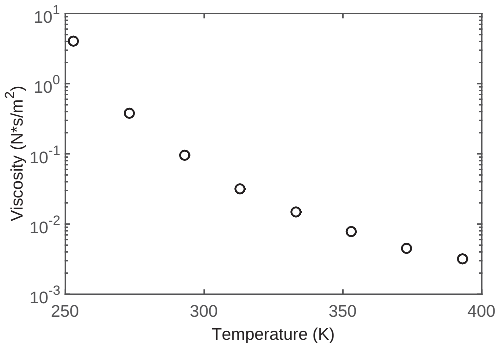
Sample Problem 8-7: Temperature dependence of viscosity

Viscosity, μ, is a property of gases and fluids that characterizes their resistance to flow. For most materials, viscosity is highly sensitive to temperature. Below is a table that gives the viscosity of SAE 10W oil at different temperatures (data from *B.R.* Munson, D.F. Young, and T.H. Okiishi, *Fundamentals of Fluid Mechanics*, 4th ed., John Wiley and Sons, 2002). Determine an equation that can be fitted to the data.

T (°C)	–20	0	20	40	60	80	100	120
μ (N s/m²) ($\times 10^{-5}$)	4	0.38	0.095	0.032	0.015	0.007 8	0.004 5	0.003 2

Solution

To determine what type of equation might provide a good fit to the data, μ is plotted as a function of *T* (absolute temperature) with a linear scale for *T* and a logarithmic scale for μ. The plot, shown on the right, indicates that the data points do not appear to line up along a straight line. This means that a simple exponential function of the form $y = be^{mx}$, which models a straight line with these axes, will not provide the best fit. Since the points in the figure appear to lie along a curved line, a function that can possibly have a good fit to the data is:



$$\ln(\mu) = a_2T^2 + a_1T + a_0$$

This function can be fitted to the data by using MATLAB's `polyfit(x,y,2)` function (second-degree polynomial), where the independent variable is *T* and the dependent variable is ln(μ). The equation above can be solved for μ to give the viscosity as a function of temperature:

$$\mu = e^{(a_2T^2 + a_1T + a_0)} = e^{a_0}e^{a_1T}e^{a_2T^2}$$

The following program determines the best fit to the function and creates a plot that displays the data points and the function.

```
T=[-20:20:120];
mu=[4 0.38 0.095 0.032 0.015 0.0078 0.0045 0.0032];
TK=T+273;
p=polyfit(TK,log(mu),2)
```

```
Tplot=273+[-20:120];
muplot = exp(p(1)*Tplot.^2 + p(2)*Tplot + p(3));
semilogy(TK,mu,'o',Tplot,muplot)
```

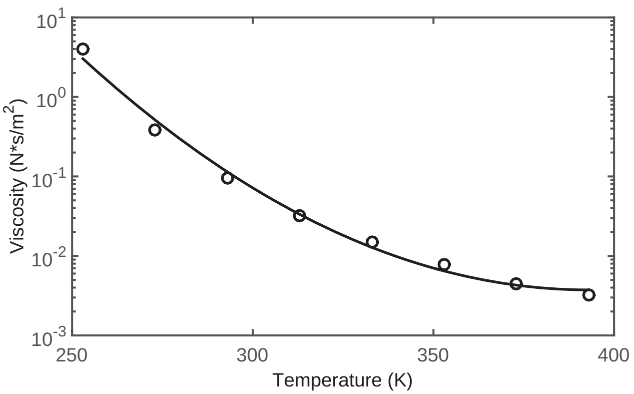
When the program executes (saved as Chap8SamPro7), the coefficients that are determined by the `polyfit` function are displayed in the Command Window (shown below) as three elements of the vector `p`.

```
>> Chap8SamPro7
p =
    0.0003   -0.2685   47.1673
```

With these coefficients the viscosity of the oil as a function of temperature is:

$$\mu = Ae^{(0.0003T^2 - 0.2685T + 47.1673)} = e^{47.1673}e^{-0.2685T}e^{0.0003T^2}$$

The plot that is generated shows that the equation correlates well to the data points (axis labels were added with the Plot Editor).



8.6 PROBLEMS

1. Plot the polynomial $y = 0.9x^5 - 0.3x^4 - 15.5x^3 + 7x^2 + 36x - 7$ in the domain $-4 \le x \le 4$. First create a vector for x, next use the `polyval` function to calculate y, and then use the `plot` function.

2. Plot the polynomial $y = 0.7x^4 - 13.5x^2 + 6x - 37$ in the domain $-5 \le x \le 5$. First create a vector for x, next use the `polyval` function to calculate y, and then use the `plot` function.

3. Determine the polynomial $y(x)$ that has roots at $x = -0.7$, $x = 0.5$, $x = 3.4$, and $x = 5.8$. Make a plot of the polynomial in the domain $-1 \le x \le 6$.

4. Use MATLAB to carry out the following multiplication of two polynomials:

$$(2x^2 - 3x + 6)(-5x^3 + 4x - 7)$$

5. Use MATLAB to carry out the following multiplication of polynomials:

$$x(x + 1.8)(x - 0.4)(x - 1.6)$$

Plot the polynomial in the domain $-2 \le x \le 2$.

6. Use MATLAB to divide the polynomial
$-9x^6 + 12x^5 + 5x^4 - 9x^3 + 17x^2 - 7x - 15$ by the polynomial $3x^2 - 2x - 3$.

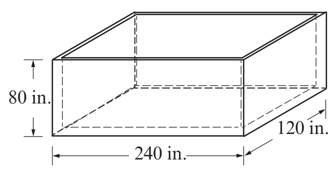
7. Use MATLAB to divide the polynomial
$0.9x^5 - 5.96x^4 + 20.85x^3 - 24.1x^2 + 3x + 7.5$ by the polynomial
$0.5x^3 - 2.2x^2 + 6x + 3$.

8. The product of four consecutive even integers is 1,488,384. Using MATLAB's built-in function for operations with polynomials, determine the two integers.

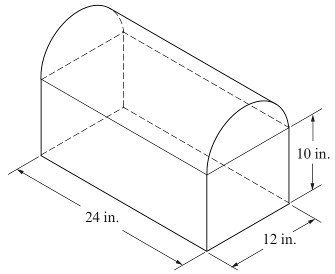
9. The product of three integers with spacing of 3 between them (e.g., 9, 12, 15) is 11,960. Using MATLAB's built-in functions for operations with polynomials, determine the three integers.

10. The product of three distinct integers is 6,240. The sum of the numbers is 85. The difference between the largest and the smallest is 57. Using MATLAB's built-in functions for operations with polynomials, determine the three integers.

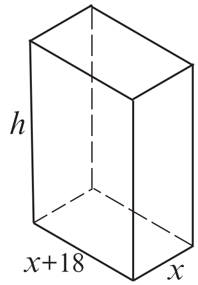
11. A rectangular steel container (no top) has the outside dimensions shown in the figure. The thickness of the bottom surface is t, and the thickness of side walls is $t/2$. Determine t if the weight of the container is 1,300 lb. The specific weight of steel is 0.284 lb/in.^{3}.



12. An aluminum container has the geometry shown in the figure (the bottom part is a rectangular box and the top is half a cylinder). The outside dimensions are shown. The wall thickness of the bottom and all the vertical walls is $2t$, and the walls thickness of the cylindrical section is t. Determine t if the tank weight is 30 lb. The specific weight of aluminum is 0.101 lb/in.^{3}.



13. A rectangular box (no top) is welded together using sheet metal. The length of the box's base is 18 in. longer than its width. The total surface area of the sheet metal that is used is 2,500 in.^{2}.
    (*a*) Using polynomials write an expression for the volume V in terms of x.
    (*b*) Make a plot of V versus x for $5 \le x \le 35$ in.
    (*c*) Determine the dimensions of the box that maximizes the volume and determine that volume.



14. The probability P of selecting three distinct numbers out of n numbers is calculated by:

$$P = \frac{2 \cdot 3}{n(n-1)(n-2)}$$

Determine how many numbers, n, should be in a lottery game such that the probability of matching three numbers out of n numbers will be at least 1/100,000, but not greater than 1/95,000.

15. Write a user-defined function that adds or subtracts two polynomials of any order. Name the function `p=polyadd(p1,p2,operation)`. The first two input arguments `p1` and `p2` are the vectors of the coefficients of the two polynomials. (If the two polynomials are not of the same order, the function adds the necessary zero elements to the shorter vector.) The third input argument `operation` is a string that can be either `'add'` or `'sub'`, for adding or subtracting the polynomials, respectively, and the output argument is the resulting polynomial.

    Use the function to add and subtract the following polynomials:

$f_1(x) = 8x^6 + 10x^5 - 5x^3 + 13x^2 - 4x - 2$ and $f_2(x) = 4x^4 + 7x^2 + 6$

16. Write a user-defined function that multiplies two polynomials. Name the function `p=polymult(p1,p2)`. The two input arguments `p1` and `p2` are vectors of the coefficients of the two polynomials. The output argument `p` is the resulting polynomial.

    Use the function to multiply the following polynomials:

$f_1(x) = -2x^6 + 3x^4 + 4x^3 - 7x + 8$ and $f_2(x) = 5x^4 - 4x^2 + 3x - 5$
Check the answer with MATLAB's built-in function `conv`.

17. Write a user-defined function that calculates the maximum (or minimum) of a quadratic equation of the form:

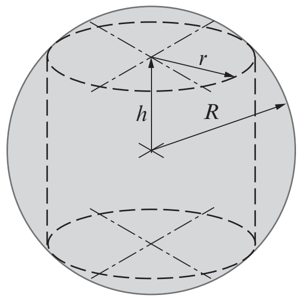
$$f(x) = ax^2 + bx + c$$

Name the function `[x,y,w] = maxormin(a,b,c)`. The input arguments are the coefficients `a`, `b`, and `c`. The output arguments are `x`, the coordinate of the maximum (or minimum); `y`, the maximum (or minimum) value; and `w`, which is equal to 1 if `y` is a maximum and equal to 2 if `y` is a minimum.

Use the function to determine the maximum or minimum of the following functions:

(*a*) $f(x) = 3x^2 - 7x + 14$ (*b*) $f(x) = -5x^2 - 11x + 15$

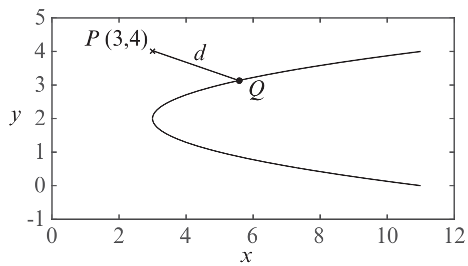
18. A cylinder with base radius *r* and height *h* is constructed inside a sphere such that it is in contact with the surface of a sphere, as shown in the figure. The radius of the sphere is $R = 11$ in.



(*a*) Create a polynomial expression for the volume *V* of the cylinder in terms of *h*.
(*b*) Make a plot of *V* versus *h* for $0 \le h \le 11$ in.
(*c*) Using the `roots` command determine *h* if the volume of the cylinder is 2,000 in.^{3}.
(*d*) Determine the value of *h* that corresponds to the cylinder with the largest possible volume, and determine that volume.

19. Consider the parabola:

$x = 2(y-2)^2 + 3$, and the point $P(3, 4)$.



(*a*) Write a polynomial expression for the distance *d* from point *P* to an arbitrary point *Q* on the parabola.
(*b*) Make a plot of *d* versus *y* for $0 \le y \le 4$.
(*c*) Determine the coordinates of *Q* if $d = 3$ (there are two points).
(*d*) Determine the coordinates of *Q* that correspond to the smallest *d*, and calculate the corresponding value of *d*.
(*e*) Make a plot that shows the parabola, point *P*, the two points from part (*c*), and the point from part (*d*).

20. The following data is given:

x	–5	–4	–1	1	4	6	9	10
y	12	10	6	2	–3	–6	–11	–12

(*a*) Use linear least-squares regression to determine the coefficients m and b in the function $y = mx + b$ that best fits the data.
(*b*) Make a plot that shows the function and the data points.

21. The boiling temperature of water T_B at various altitudes h is given in the following table. Determine a linear equation in the form $T_B = mh + b$ that best fits the data. Use the equation for calculating the boiling temperature at 5,000 m. Make a plot of the points and the equation.

h (ft)	–1,000	0	3,000	8,000	15,000	22,000	28,000
T (°F)	213.9	212	206.2	196.2	184.4	172.6	163.1

22. The U.S. population in selected years between 1815 and 1965 is listed in the table below. Determine a quadratic equation in the form $P = a_2t^2 + a_1t + a_0$, where t is the number of years after 1800 and P is the population in millions, that best fits the data. Use the equation to estimate the population in 1915 (the population was 98.8 millions). Make a plot of the population versus the year that shows the data points and the equation.

Year	1815	1845	1875	1905	1935	1965
Population (millions)	8.3	19.7	44.4	83.2	127.1	190.9

23. The number of bacteria N_B measured at different times t is given in the following table. Determine an exponential function in the form $N_B = Ne^{\alpha t}$ that best fits the data. Use the equation to estimate the number of bacteria after 5 h. Make a plot of the points and the equation.

t (h)	0	1	3	4	6	7	9
N_B	500	600	1,000	1,400	2,100	2,700	4,100

24. Growth data of a sunflower plant is given in the following table:

Day	7	21	35	49	63	77	91
Height (in.)	8.5	21	50	77	89	98	99

The data can be modeled with a function in the form $y = \frac{H}{1+e^{-(a+bt)}}$, where y is the height, H is a maximum height, a and b are constants, and t is the number of days. By using the method described in Section 8.2.2, and assuming that $H = 102$ in., determine the constants a and b such that the function best fits the data. Use the function to estimate the height in day 40. In one figure, plot the function and the data points.

25. Use the growth data from Problem 24 for the following:
    (*a*) Curve-fit the data with a third-order polynomial. Use the polynomial to estimate the height in day 40.
    (*b*) Fit the data with linear and spline interpolations and use each interpolation to estimate the height in day 40.

    In each part make a plot of the data points (circle markers) and the fitted curve or the interpolated curves. Note that part (*b*) has two interpolation curves.

26. The following points are given:

x	1	2.2	3.7	6.4	9	11.5	14.2	17.8	20.5	23.2
y	12	9	6.6	5.5	7.2	9.2	9.6	8.5	6.5	2.2

    (*a*) Fit the data with a first-order polynomial. Make a plot of the points and the polynomial.
    (*b*) Fit the data with a second-order polynomial. Make a plot of the points and the polynomial.
    (*c*) Fit the data with a third-order polynomial. Make a plot of the points and the polynomial.
    (*d*) Fit the data with an fifth-order polynomial. Make a plot of the points and the polynomial.

27. The standard air density, D (average of measurements made), at different heights, h, from sea level up to a height of 33 km is given below.

h (km)	0	3	6	9	12	15
D (kg/m^3)	1.2	0.91	0.66	0.47	0.31	0.19
h (km)	18	21	24	27	30	33
D (kg/m^3)	0.12	0.075	0.046	0.029	0.018	0.011

    (*a*) Make the following four plots of the data points (density as a function of height): (1) both axes with linear scale; (2) h with log axis, D with linear axis; (3) h with linear axis, D with log axis; (4) both log axes. According to the plots, choose a function (linear, power, exponential, or logarithmic) that best fits the data points and determine the coefficients of the function.

(*b*) Plot the function and the points using linear axes.

28. Write a user-defined function that determines the best fit of an exponential function of the form $y = be^{mx}$. Name the function `[b,m] = expofit(x,y)`, where the input arguments `x` and `y` are vectors with the coordinates of the data points, and the output arguments `b` and `m` are the constants of the fitted exponential equation. Use `expofit` to fit the data below. Make a plot that shows the data points and the function.

x	0.4	2.2	3.1	5.0	6.6	7.6
y	1.7	10.1	26.9	61.2	158	398

29. Estimated values of thermal conductivity of silicon at different temperatures are given in the following table.

T (K)	2	4	6	8	10	20	40	60
k (W/m-K)	46	300	820	1,560	2,300	5,000	3,500	2,100
T (K)	80	100	150	250	350	500	1,000	1,400
k (W/m-K)	1,350	900	400	190	120	75	30	20

(*a*) Make a plot of k versus T using log scale on both axes.

(*b*) Curve-fit the data with a second-order polynomial $y = ax^2 + bx + c$ in which $x = \log(T)$ and $y = \log(k)$. Once the coefficients a, b, and c are determined, write an equation for k as a function of $\log(T)$. Use this equation for curve-fitting the data. Make a second plot that shows the data points with markers and the curve-fitted equation with a solid line.

(*c*) Repeat part (*b*) using a third-order polynomial.

30. Measurements of the concentration, C, of a substance during a chemical reaction at different times t are shown in the table.

t (h)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
C (g/L)	1.7	3.1	5.7	9.1	6.4	3.7	2.8	1.6	1.2	0.8	0.7	0.6

(*a*) Suppose that the data can be modeled with an equation in the form:

$$C(t) = \frac{1}{a_2 t^2 + a_1 t + a_0}$$

Determine the coefficients a_0, a_1, and a_2 such that the equation best fits the data. Use the equation to estimate the concentration at $t = 2$ h.

Make a plot of the data points and the equation.

(*b*) Suppose that the data can be modeled with an equation in the form:

$$C(t) = \frac{1}{a_3t^3+a_2t^2+a_1t+a_0}$$

Determine the coefficients a_0, a_1, a_2, and a_3 such that the equation best fits the data. Use the equation to estimate the concentration at $t = 2$ h. Make a plot of the data points and the equation.

31. Use the data from Problem 30 for the following:

(*a*) Fit the data with linear interpolation. Estimate the concentration at $t = 2.25$. Make a plot that shows the data points and curve made of interpolated points.

(*b*) Fit the data with spline interpolation. Estimate the concentration at $t = 2.25$ h. Make a plot that shows the data points and a curve made of interpolated points.

32. The relationship between two variables y and x is known to be:

$$y = a\frac{x}{b+x}$$

The following data points are given:

x	5	10	15	20	25	30	35	40	45	50
y	15	25	32	33	37	35	38	39	41	42

Determine the constants a and b by curve-fitting the equation to the data points. Make a plot of y versus x. In the plot show the data points with markers and the curve-fitted equation with a solid line. Use the equation to estimate y at $x = 23$. (The curve fitting can be done by writing the reciprocal of the equation and using a first-order polynomial.)

33. Curve-fit the data from the previous problem with a third-order polynomial. Use the polynomial to estimate y at $x = 23$. Make a plot of the points and the polynomial.

34. When rubber is stretched, its elongation is initially proportional to the applied force, but as it reaches about twice its original length, the force required to stretch the rubber increases rapidly. The force, as a function of elongation, that was required to stretch a rubber specimen that was initially 3 in. long is displayed in the following table.

(*a*) Curve-fit the data with a fourth-order polynomial. Make a plot of the data points and the polynomial. Use the polynomial to estimate the force when the rubber specimen was 11.5 in. long.

(*b*) Fit the data with spline interpolation (use MATLAB's built-in function `interp1`). Make a plot that shows the data points and a curve made by interpolation. Use interpolation to estimate the force when the rubber specimen was 11.5 in. long.

Force (lb)	0	0.6	0.9	1.16	1.18	1.19	1.24	1.48
Elongation (in.)	0	1.2	2.4	3.6	4.8	6.0	7.2	8.4
Force (lb)	1.92	3.12	4.14	5.34	6.22	7.12	7.86	8.42
Elongation (in.)	9.6	10.8	12.0	13.2	14.4	15.6	16.8	18

35. The transmission of light through a transparent solid can be described by the equation:

$$I_T = I_0(1-R)^2 e^{-\beta L}$$

where I_T is the transmitted intensity, I_0 is the intensity of the incident beam, β is the absorption coefficient, L is the length of the transparent solid, and R is the fraction of light which is reflected at the interface. If the light is normal to the interface and the beams are transmitted through air, $R = \left(\frac{n-1}{n+1}\right)^2$ where n is the index of refraction for the transparent solid. Experiments measuring the intensity of light transmitted through specimens of a transparent solid of various lengths are given in the following table. The intensity of the incident beam is 5 W/m^2.

L (cm)	0.5	1.2	1.7	2.2	4.5	6.0
I_T (W/m^2)	4.2	4.0	3.8	3.6	2.9	2.5

Use this data and curve fitting to determine the absorption coefficient and index of refraction of the solid.