Foundation of Mathematics1

CHAPTER 1 LOGIC THEORY

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Contents

Chapter 1	Logic Theory	Logic, Truth Table, Tautology, Contradiction, Contingency, Rules of Proof , Logical Implication, Quantifiers, Logical Reasoning, Mathematical Proof.
Chapter 2	Sets	Definitions, Equality of Sets, Set Laws
Chapter 3	Relations on Set	Cartesian Product, Relations.
Chapter 4	Algebra of Mappings	Mappings, Types of Mappings, Composite Mapping.

References

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THE GREEK ALPHABET

letter	name	capital
α	Alpha	Α
β	Beta	Β
γ	Gamma	Γ
δ	Delta	Δ
ε	Epsilon	Ε
ζ	Zeta	Ζ
η	Eta	Η
θ	Theta	Θ
ι	Iota	Ι
κ	Kappa	Κ
λ	Lambda	Λ
μ	Mu	Μ
ν	Nu	Ν
ξ	Xi	Ξ
ο	Omicron	Ο
π	Pi	Π
ρ	Rho	Ρ
σ ς	Sigma	Σ
τ	Tau	Τ
υ	Upsilon	Υ
φ	Phi	Φ
χ	Chi	Χ
ψ	Psi	Ψ
ω	Omega	Ω

Chapter One

Logic Theory

1.1. Logic

Definition 1.1.1.

(i) Logic is the theory of systematic reasoning, and symbolic logic is the formal theory of logic.

(ii)A **logical proposition (statement or formula)** is a declarative sentence that is either true (denoted either T or 1) or false (denoted either F or 0) but not both.

(iii) The truth or falsehood of a logical proposition is called its **truth value.**

Notation: Variables are used to represent logical propositions. The most common variables used are p, q, and r.

Example 1.1.2.

$x + 2 = 2x$ when $x = -2$.

All cars are brown.

$2 \times 2 = 5$.

Here are some sentences that are not logical propositions (**paradox**).

Look out! **(Exclamatory)**

How far is it to the next town? **(Interrogative)**

$x + 2 = 2x$.

“Do you want to go to the movies?”**(Interrogative)**

“Clean up your room.” **(Imperative)**

1.2. Truth Table

1.2.1. What is a Truth Table?

(i) A truth table is a tool that helps you analyze statements or arguments (defined later) in order to verify whether or not they are logical, or true.

(ii)A truth table of a logical proposition shows the condition under which the logical proposition is true and those under which it is false.

1.2.2. There are six basic operations called **connectives** that will utilize when creating a truth table. These operations are given below.

English Name	Math Name	Symbol
"and"	Conjunction	$\wedge$
"or"	Disjunction	$\vee$
"Exclusive"= "or but not both"	xor	$\underline{\vee}$
"if … then"	Implication	$\rightarrow$
"if and only if "	equivalence	$\leftrightarrow$
"not"	Negation	$\sim$

Definition 1.2.3. (Compound Statements)

If two or more logical propositions compound by connectives called compound proposition (statement).The truth value of a compound proposition depends only on the value of its components.

The rules for these connectives (operations) are as follows:

AND ($\wedge$) (conjunction): these statements are true only when both p and q are true.

AND $\wedge$ (Conjunction)		
p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

OR (∨) (disjunction): these statements are false only when both p and q are false.

OR ∨ (Disjunction)		
p	q	p∨ q
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Exclusive(⊻) one of p or q (read p or else q)

⊻ (Exclusive)		
p	q	p⊻ q
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

If → Then Statements – These statements are false only when p is true and q is false (because anything can follow from a false premise).

If → Then		
p	q	p→ q
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Here, p called **hypothesis (antecedent)** and q called **consequent (conclusion)**.

- Equivalent Forms of (**p→ q)** read as:

1- If p then q":
2- p implies q
3- p is a sufficient condition for q

(Existence of H_2O is sufficient to exist of Oxygen(O))
4- p only if q=if not **q** then not **p**.
5- q if p.
6- q whenever p
7- q is a necessary condition for p.
(Existence of O is necessary to exist of H_2O)
8- q follows from p.
9-q, provided that p.

To understand why the conditional statement is false only in the case when p is true but q is false considering the following example:

- Suppose your dad promises you:

"If you get an A in Foundation1, then I will buy you a laptop".

Here, p is "you get an **A** in Foundation1",
q is "I will buy you a laptop".

Then the only situation you can accuse your dad of breaking his promise is when

you get an A in Foundation1
but (and)
your dad does not buy you a laptop.

If you do not get an **A** in Foundtation1, then whether you dad buys you a laptop or not, you can't say that he breaks his promise.

- The statement q →p is called the **converse** of the statement p →q and the statement ~p →~q is called the **inverse**.

For instance "**if Ali is from Baghdad then Ali is from Iraq"** is true, but the converse **"if Ali is from Iraq then Ali is from Baghdad"** may be false. The inverse "**if Ali is not from Baghdad then Ali is not from Iraq"** may be false.

- **Note** that **t**he statements **p→ q** and **q→ p** are different.

If and only If Statements – These statements are true only when both p and q have the same truth (logical) values.

If ↔ Then		
p	q	p↔ q
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

NOT ~ (negation) The "not" is simply the opposite or complement of its original value.

NOT ~ (negation)	
P	~p
T	**F**
F	**T**

- Note that, the negation is meaningful when used with only one logical proposition. This is not true of the other connectives.

Examples 1.2.4. Write the following statements symbolically, and then make a truth table for the statements.

(i) If I go to the mall or go to the stadium, then I will not go to the gym.

(ii) If the fish is cooked, then dinner is ready and I am hungry.

<u>Solution.</u>

(i) Suppose we set

p = I go to the mall

q = I go to the stadium

r = I will go to the gym

The proposition can then be expressed as "If p or q, then not r," or $(p \vee q) \rightarrow \sim r$.

p	q	r	p∨ q	~r	(p ∨ q) →~r
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

(ii) Suppose we set

f = the fish is cooked.
r = dinner is ready.
h = I am hungry.

(a) $f \to (r \wedge h)$

(b) $(f \to r) \wedge h$

f	r	h	r ∧ h	f→(r ∧ h)	f →r	(f→r) ∧ h
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Exercise 1.2.5. Build a truth table for $p \to (q \to r)$ and $(p \wedge q) \to r$.

1.3. Tautology /Contradiction / Contingency

Definition 1.3.1. (Tautology)

A tautology (theorem or lemma) is a logical proposition that is always true.

Remark 1.3.2. One informal way to check whether or not a certain logical formula is a theorem is to construct its truth table.

Example 1.3.3. $p \vee \sim p$.

Definition 1.3.4. (Contradiction)

A contradiction is a logical proposition that is always false.

Example 1.3.5. $p \wedge \sim p$.

Definition 1.3.6. (Contingency)

A contingency is a logical proposition that is neither a tautology nor a contradiction.

Example 1.3.7.

(i) The logical proposition $(p \vee q) \to \sim r$ is a contingency. See Example 1.2.4(i).

(ii) The logical proposition $p \vee \sim (p \wedge q)$ is a tautology.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercise 1.3.8.

(i) Build a truth table to verify that the logical proposition

$$(p \leftrightarrow q) \wedge (\sim p \wedge q)$$

is a contradiction.

(ii) (Low of Syllogism) Show that the logical proposition

$$[(p \to q)\wedge(q\to r)]\to(p \to r)$$

is a tautology.

Definition 1.3.9. (Logically equivalent)

Propositions **r** and **s** are logically equivalents if the truth tables of **r** and **s** are the same and denoted by **r ≡ s.**

Example 1.3.10. Show that

$$\sim(p \to q)\equiv p\wedge\sim q.$$

Solution.

Show the truth values of both propositions are identical.

p	q	$\sim q$	$p \to q$	$\sim(p \to q)$	$p\wedge\sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Remark 1.3.11. (Relation Between Logical Equivalent and Tautology)

$$(r \equiv s)\equiv(r \leftrightarrow s) \text{ is a tautology.}$$

Solution.

r	s		r	s	$r \leftrightarrow s$	
T	T	**r ≡s**	T	T	T	←
T	F		T	F	F	
F	T		F	T	F	
F	F	**r ≡s**	F	F	T	←

From the above table of the propositions r≡s and (r ↔ s is a tautology) we get that they have the same truth table.

1.3.12. Algebra of Logical Proposition

The logical equivalences below are important equivalences that should be memorized.

1-Identity Laws: $p \wedge T \equiv p$.
$p \vee F \equiv p$.

2-Domination Laws: $p \vee T \equiv T$.
$p \wedge F \equiv F$.

3-Idempotent Laws: $p \vee p \equiv p$.
$p \wedge p \equiv p$.

4- Double Negation Law: $\sim(\sim p) \equiv p$.

5- Commutative Laws: $p \vee q \equiv q \vee p$.
$p \wedge q \equiv q \wedge p$.

6- Associative Laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

7- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

8- De Morgan's Laws: $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

9- Absorption Laws: $p \wedge (p \vee q) \equiv p$.
$p \vee (p \wedge q) \equiv p$.
$p \wedge (\sim p \vee q) \equiv p \wedge q$.
$p \vee (\sim p \wedge q) \equiv p \vee q$.

10-Implication Law: $(p \rightarrow q) \equiv (\sim p \vee q)$.
11- Contrapositive Law: $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.
12- Tautology: $p \vee \sim p \equiv T$.
13- Contradiction: $p \wedge \sim p \equiv F$.
14- Equivalence: $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$.

15- $p \veebar q \equiv (p \vee q) \wedge \sim (p \wedge q)$.

Solution.

(8) We are using truth table to prove $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(14) We are using truth table to prove $(p \to q) \wedge (q \to p) \equiv (p \leftrightarrow q)$.

p	q	$p \to q$	$q \to p$	$p \to q \wedge q \to p$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

(15) $p \veebar q \equiv (p \vee q) \wedge \sim (p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$\mathbf{p \veebar q}$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F

1.4. Rules of Proof

1.4.1.

(i) Rule of Replacement.

Any term in a logical formula may be replaced by an equivalent term.

For instance, if $q \equiv r$, then $(p \wedge q) \equiv (p \wedge r)$ Rep(q:r).

(ii) Rule of Substitution.

A sentence which is obtained by substituting logical propositions for the terms of a theorem is itself a theorem.

For instance, $(p \to q) \vee w \equiv w \vee (p \to q)$ Sub(p: $p \to q$), in Commutative Law

$p \vee w \equiv w \vee p$.

(iii) Rule of Inference.

1-	$\begin{array}{c} p \\ p \to q \\ \hline \therefore q \end{array}$	6-	$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$
2-	$\begin{array}{c} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$	7-	$\begin{array}{c} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$
3-	$\begin{array}{c} p \\ \hline \therefore p \vee R \end{array}$	8-	$\begin{array}{c} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$
4-	$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$	9-	$\begin{array}{c} p \to q \\ r \to t \\ \hline \therefore p \vee r \to q \vee t \end{array}$
5-	$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$	10-	$\begin{array}{c} p \\ q \to r \\ \hline \therefore p \vee q \to p \vee r \end{array}$

Example 1.4.2.

(i) Given

(1) $p \wedge q$

(2) $p \to \sim (q \wedge r)$

(3) $s \to r$

$\therefore \sim s$

Solution:

1- $p \wedge q$	1^{st} hypothesis (premise)
2- p	Inf. (1) Properties of $\wedge$
3- q	Inf. (1) Properties of $\wedge$
4- $p \to \sim (q \wedge r)$	2^{nd} hypothesis(premise)
5- $\sim (q \wedge r)$	Inf. (2),(4)
6- $\sim q \vee \sim r$	De Morgan's Law on (5)
7- $\sim r$	Inf. (3),(6) and Domination Laws
8- $s \to r$	3^{rd} hypothesis (premise)
9- $\sim r \to \sim s$	Contrapositive Law
10- $\sim s$	Inf. (7),(9)

(ii) Given

(1) $\sim (p \vee q) \to r$

(2) $\sim p$

(3) $\sim r$

$\therefore q$

Solution:

1- $\sim(p \vee q) \rightarrow r$ 1st hypothesis (premise)

2- $\sim r$ 3rd hypothesis (premise)

3- $\sim r \rightarrow (p \vee q)$ Contrapositive Law and Double Negation Law

4- $p \vee q$ Inf. (2),(3)

5- $\sim p$ 2nd hypothesis (premise)

6- q Inf. (4),(5)

(iii) Given

(1) $\sim p \rightarrow (r \wedge s)$

(2) $p \rightarrow q$

(3) $\sim q$

$\therefore r$

Solution:

1- $p \rightarrow q$ 2nd hypothesis (premise)

2- $\sim q \rightarrow \sim p$ Contrapositive Law on (1)

3- $\sim q$ 3rd hypothesis (premise)

4- $\sim p$ Inf. (2),(3)

5- $\sim p \rightarrow (r \wedge s)$ 1st hypothesis(premise)

6- $r \wedge s$ Inf. (4),(5)

7- r Inf. (6) Properties of $\wedge$

(iv) Given

(1) $p \to (\sim r \wedge \sim s)$

(2) $p \vee \sim q$

(3) s

$\therefore \sim q \wedge s$

Solution:

1- $p \to (\sim r \wedge \sim s)$	1st hypothesis (premise)
2- $(r \vee s) \to \sim p$	Contrapositive Law on (1)
3- $p \vee \sim q$	2nd hypothesis (premise)
4- $\sim p \to \sim q$	Implication Law on (3)
5- $(r \vee s) \to \sim q$	Inf. (2),(4)
6- s	3rd hypothesis(premise)
7- $r \vee s$	Inf. (6)
8- $\sim q$	Inf. (5),(7)
9- $\sim q \wedge s$	Inf. (6),(8)

(v) Given

(1) $p \vee q$

(2) $q \to r$

(3) $\sim r$

$\therefore p$

Solution:

1- $q \to r$	2^{nd} hypothesis(premise)
2- $\sim r \to \sim q$	Contrapositive Law on (1)
3- $\sim r$	3^{rd} hypothesis(premise)
4- $\sim q$	Inf. (2),(3)
5- $p \vee q$	1^{st} hypothesis(premise)
6- $(p \vee q) \wedge \sim q$	Inf. (4),(5)
7- $(p \wedge \sim q) \vee (q \wedge \sim q)$	Distributive Law on (6)
8- $(p \wedge \sim q) \vee F$	Contradiction Law (7)
9- $(p \wedge \sim q)$	Identity Law on (8)
10- p	Inf. (9) properties of $\wedge$

(vi) Given
(1) If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on;
(2) If the sailing race is held, then the cup will be awarded;
(3) The cup was not awarded.
Does this imply that: "It rained"?
Solution.
p: rain;
q: foggy;
r: the sailing race will be held;
s: the lifesaving demonstration will go on;
t: then the cup will be awarded.

Symbolically, the proposition is

$$\begin{array}{lr}
(1) & \sim p \vee \sim q \;\rightarrow\; r \wedge s \\
(2) & r \;\rightarrow\; t \\
(3) & \sim t \\
\hline
 & p
\end{array}$$

1. $\sim t$	3^{rd} hypothesis
2. $r \rightarrow t$	2^{nd} hypothesis
3. $\sim t \rightarrow \sim r$	Contrapositive of 2
4. $\sim r$	Inf. (1),(3)
5. $\sim p \vee \sim q \rightarrow r \wedge s$	1^{st} hypothesis
6. $\sim(r \wedge s) \rightarrow \sim(\sim p \vee \sim q)$	Contrapositive of 5
7. $\sim r \vee \sim s \rightarrow (p \wedge q)$	De Morgan's law and double negation law on (5)
8. $\sim r \vee \sim s$	Inf. (4) and domination law
9. $p \wedge q$	Inf. (7),(8)
10. p	Inf. (9)

Example 1.4.3. Use the logical equivalences to show that
(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$,
(ii) $\sim(p \vee \sim(p \wedge q))$ is a contradiction,
(iii) $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p \wedge \sim q)$,

(iv) $p \vee (p \wedge q) \equiv p$ (Absorption Law).

Solution.

(i) $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$ Implication Law

$\equiv \sim(\sim p) \wedge \sim q$ De Morgan's Law
$\equiv p \wedge \sim q$ Double Negation Law

(ii) $\sim(p \vee \sim(p \wedge q))$
$\equiv \sim p \wedge \sim(\sim(p \wedge q))$ De Morgan's Law
$\equiv \sim p \wedge (p \wedge q)$ Double Negation Law
$\equiv (\sim p \wedge p) \wedge q$ Associative Law
$\equiv F \wedge q$ Contradiction Law
$\equiv F$ Domination Law and Commutative Law.

(iii) $\sim (p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ DeMorgan's Law

$\equiv \sim p \wedge (p \vee \sim q)$ Double Negation Law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ Distribution Law

$\equiv (p \wedge \sim p) \vee (\sim p \wedge \sim q)$ Commutative Law

$\equiv F \vee (\sim p \wedge \sim q)$ Contradiction Law

$\equiv (\sim p \wedge \sim q) \vee F$ Commutative Law

$\equiv (\sim p \wedge \sim q)$ Identity Law

(iv) $p \vee (p \wedge q)$
$\equiv (p \wedge T) \vee (p \wedge q)$ Identity Law (in reverse)
$\equiv p \wedge (T \vee q)$ Distributive Law (in reverse)
$\equiv p \wedge T$ Domination Law
$\equiv p$ Identity Law

Example 1.4.4. Find a simple form for the negation of the proposition
"If the sun is shining, then I am going to the ball game."

Solution.

p: the sun is shining q: I am going to the football game

This proposition is of the form $p \to q$. Since $\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$. This is the proposition **"The sun is shining, and I am not going to the football game."**

1.5. Logical Implication

Definition 1.5.1. (Logical implication)
We say the logical proposition **"r"** implies the logical proposition **"s"** (or **s** logically deduced from **r**) and write(**r**$\Longrightarrow$ **s**) iff (**r**$\rightarrow$**s**) is a tautology.

Example 1.5.2. Show that $[(p \rightarrow t) \wedge (t \rightarrow q)] \Longrightarrow (p \rightarrow q)$.

Solution. Let P: the proposition $(p \rightarrow t) \wedge (t \rightarrow q)$

Q: the proposition $p \rightarrow q$

p	t	q	p→t	t→ **q**	P	Q	P → Q
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Remark 1.5.3.

(i) We use $(r \Longrightarrow s)$ to imply that the statement $(r \rightarrow s)$ is true, while the statement$(r \rightarrow s)$ alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for "**implies**".

(ii) If $(r \Longrightarrow s)$ and $(s \Longrightarrow r)$, then we denote that by $(r \Longleftrightarrow s)$.

Example 1.5.4. Show that
(i)$(r \Longrightarrow s) \equiv [(\sim r \vee s)$ is tautology].
(ii)$(r \Longleftrightarrow s) \equiv (r \equiv s)$.
Solution.
(i)
1- $(r \Longrightarrow s) \equiv (r \rightarrow s)$ is tautology (Def. of $\Longrightarrow$)
2-$(r \rightarrow s) \equiv (\sim r \vee s)$ Logical Implication Law
3- $(\sim r \vee s)$ is tautology Inf. (1),(2)

(ii)

1- $\left.\begin{array}{l}(r \Longrightarrow s) \equiv (r \to s) \text{ is tautology} \\ (s \Longrightarrow r) \equiv (s \to r) \text{ is tautology}\end{array}\right\}$ Def. of $(\Longrightarrow)$ and $(\Longleftrightarrow)$

2-$(r \to s) \land (s \to r)$ is tautology.
3- $r \leftrightarrow s$ is tautology Equivalence Law
4- $r \equiv s$. Inf. Remark 1.3.11

- Generally, the statement and its converse not necessary equivalent. Therefore, $p \Longrightarrow q$ does not mean that $q \Longrightarrow p$.

Example 1.5.5. The statement “the triangle which has equal sides, has two equal legs” equivalent to the statement “ the triangle which has not two equal legs has no equal sides”.

1.6. Quantifiers

Definition 1.6.1.

(i) A **predicate** or **propositional function** is a statement (formula) containing variables and that may be true or false depending on the values of these variables.

- That is, a predicate is a property or relationship between objects represented symbolically.
- We represent a predicate by a letter followed by the variables enclosed between parenthesis: $P(x), Q(x,y)$, etc.

(ii) An ***example*** for $P(x)$ is: value of x for which $P(x)$is true.

(iii) A ***counterexample*** $P(x)$ is: value of x for which $P(x)$ is false.

(iv) The set, X which contain all possible value that satisfy the formula P is called a **universal set**.

(v) A set Y which contains all values x belong to set X such that $P(x)$ is true is called a **solution set.**

$$Y = S_P = \{x \in X: P(x) \text{is true}\}.$$

Example 1.6.2.

(i)$P(x) = x \leq 5 \wedge x > 3$ is true for $x = 4$ and false for $x = 6$ (counterexample).
(ii)$P(x) =x \leq 5 \wedge x > 3$, for every real numbers, x which is definitely false.

(iii) There exists an x such that $(x) = x \leq 5 \wedge x > 3$,"which is definitely true.

(iv) Given the statement "**Ahmad is a logician**".

Let P represent **'is a logician'** and let x represent **'Ahmad'**. The predicate form of this statement is $P(x)$. That is, $P(x)$ =Ahmad is a logician.

(v) Let r: x is married to y.

Let M represent **"married".** Then r $= M(x,y)$.

(vi) Let r: The numbers x and y are both odd.
This statement means (x is odd) $\wedge$ (y is odd).

Let P represent **'is a odd'** and let x, y represent **'numbers'**. The predicate form of this statement is $P(x) \wedge P(y)$.

Definition 1.6.3.

(i) The phrase **"for all x"** (**"for every x"**, **"for each x"**) is called a **universal quantifier** and is denoted by $\forall\, x$.

(ii) The phrase **"for some x"** (**"there exists an x"**) is called an **existential quantifier** and is denoted by $\exists\, x$.

Definition 1.6.4. (The Universal Quantifier Proposition)

Let $f(x)$ be a proposition function which depend only on x. A sentence $\forall\, x, f(x)$ read "For all x, $f(x)$" mean

"For all values x in X(universal set), the predicate $f(x)$ is true."; that is,

$$\frac{\forall x, f(x)}{\therefore\ f(a)}$$

Example 1.6.5.

(i) r: The square of all real numbers are positive.

$$\boxed{\text{r}: \forall x \in \mathbb{R}, \qquad (x^2 > 0).}$$

(ii) r: The commutative law of addition of real numbers is holed.

$$\boxed{\text{r}: \forall\, x, \forall\, y \in \mathbb{R}, \qquad (x + y) = (y + x).}$$

(iii) r: The associative law of addition of real numbers is holed.

$$\boxed{\text{r}: \forall\, x, \forall\, y, \forall\, z \in \mathbb{R}, \big((x + y) + z = x + (y + z)\big).}$$

(iv) r: All logicians are exceptional.

Let L represent **'set of logician'** and let E represent **'is exceptional'**. The predicate form of this statement is $\boxed{\text{r}: \forall x \in L,\ \ E(x)}$.

(v) r: All cars are red.
Let $X :=$ **Set of cars**, $f :=$ **is red**. Then, r : $\forall x \in X, f(x)$.

Remark 1.6.6.

(i) The *"***all***"* form the universal quantifier, is frequently encountered in the following context: $\forall x(f(x) \longrightarrow Q(x))$,

which may be read,

"For all x in a universal set X satisfying $f(x)$ also satisfy $Q(x)$".

For example:

(a) r: All logicians are exceptional.
Let L represent **'is a logician'** and let E represent **'is an exceptional'**. Then

- Predicate Logic: r:$\forall x(L(x) \longrightarrow E(x))$
- In logical English: "For all x, if x is a logician, then x is exceptional".

(b) r: The square of all real numbers are positive.
Let P represent: $\in \mathbb{R}$ and let Q represent **"square is positive"**.

- Predicate Logic: r:$\forall x(P(x) \longrightarrow Q(x))$; that is,
  $\boxed{\text{r: } \forall x(\text{ if } x \in \mathbb{R} \longrightarrow (x^2 > 0).}$
- In logical English: "For all x, if x is real number, then x^2 is positive."

(c) Every (each, any) integer number is even (or: Integer numbers are even).

Let P represent: $\in \mathbb{Z}$ and let E represent**" is even"**.

- Predicate Logic: r:$\forall x(P(x) \longrightarrow E(x))$; that is,
  $\boxed{\text{r: } \forall x(\text{ if } x \in \mathbb{Z} \longrightarrow E(x)).}$
- In logical English: "For all x, if x is an integer, then x is even."

(ii) Parentheses are crucial here; be sure you understand the difference between the "all" form and $\boxed{\forall x, f(x) \longrightarrow \forall x, Q(x)}$ and $\boxed{(\forall x, f(x)) \longrightarrow Q(x)}$.

Definition 1.6.7. (The Existential Quantifier Proposition)

A sentence $\exists x, f(x)$ read "For some x, $f(x)$" or " For some x such that $f(x)$" mean "For some $x \in X$ (universal set), the predicate $f(x)$ is true"; that is,

$$\frac{f(a)}{\therefore \exists x, f(x)}.$$

Example 1.6.8.

(i)$\exists\ x{:}\ (x\ \geq\ x^2)$ is true since $x = 0$ is a solution. There are many others.

(ii) r: Some logicians are exceptional.
Let L represent **'set of logician'** and let E represent **'is exceptional'**. The predicate form of this statement is $\boxed{\text{r:}\exists x{\in}L,\ \ E(x)}$.

(iii) r: There is a car which is red.
Let $X \coloneqq$ **Set of cars**, $f{:}=$**is red**. Then, r :$\exists x \in X, f(x)$.

Remark .1.6.9.

(i) The "**some**" form the existential quantifier, is frequently encountered in the following context: $\exists x(f(x)\ \wedge\ Q(x)),$

which may be read,

"Some x in a universal set X satisfying $f(x)$ and satisfy $Q(x)$".

For example:

(a) r: Some logicians are exceptional.
Let L represent **'is a logician'** and let E represent **'is exceptional'**. Then

- Predicate Logic: r:$\exists x(L(x) \wedge\ E(x))$
- In logical English: "For some$x$, x is a logician and x is exceptional."

(b) r: The square of some integers numbers are four (or: There is an integer for which its square is four)
Let P represent: $\in \mathbb{Z}$ and let Q represent **" is 4"**.

- Predicate Logic: r:$\exists x(P(x) \wedge\ Q(x))$; that is,
  $\boxed{\text{r: } \exists x(\ x \in \mathbb{Z}\ \ \ \wedge\ \ \ x^2 = 4).}$
- In logical English: "For some x, $x$is an integer number and $x^2 = 4$".

(c) At least one integer number is even (or: Some integers are even).

Let P represent: $\in \mathbb{Z}$ and let E represent **" is even"**.

- Predicate Logic: r:$\exists x(P(x) \wedge\ E(x))$; that is,
  $\boxed{\text{r: } \exists x(x \in \mathbb{Z} \wedge\ E(x)).}$
- In logical English: "For some x, x is an integer number and x is even."

Negation Rules of Quantifiers 1.6.10.
(i) When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negative the statement.
(ii) $\sim(x = y) = (x \neq y)$.
(iii) $\sim(x \equiv y) = (x \not\equiv y)$.
(iv) $\sim(x < y) = (y \leq x)$.
(v) $\sim(x \in Y) = (x \notin Y)$.
(vi) ~ (Even number) = Odd number.

Now define the a formal universal quantifier proposition using negation.

Definition 1.6.11.

(i) $\forall x, f(x) = (\sim\exists\,)x, \sim f(x)$.
(ii) $\exists\, x, f(x) \equiv (\sim\forall)x, \sim f(x)$.

Example 1.6.12.
r: All logicians are exceptional.

Let L represent **'set of logician'** and let E represent **'is exceptional'**.

- Predicate Logic: $\boxed{\text{r:}\forall x{\in}L,\ \ E(x) = \sim\exists\, x, \sim E(x)}$.
- In logical English: "There is no x is a logician, for which x is not exceptional."

Equivalent Definitions 1.6.13.
(i) $\sim\big(\forall x, f(x)\big) \equiv\ \exists\, x, \sim f(x)$.
(ii)$\sim\big(\exists\, x, f(x)\big) \equiv \forall\, x, \sim f(x)$.
(iii)$\sim\big[\forall x\big(f(x) \longrightarrow\ Q(x)\big)\big] \equiv \exists x\big(f(x) \wedge \sim Q(x)\big)$
$\equiv$ Some $f(x)$ are not $Q(x)$
(iv)$\sim\Big(\exists\, x, \big(f(x) \wedge Q(x)\big)\Big) \equiv \forall\, x, \sim f(x) \vee \sim Q(x) \equiv \forall x(f(x)\ \longrightarrow \sim Q(x))$
$\equiv$ No $f(x)$ are $Q(x)$

Example1.6.14.

(i) Express each of the following sentences in the form $\forall x, P(x)$ and then give its negation in both cases $\forall x, P(x)$ and in words.

r:The square of every real number is non-negative.

Solution.

- $\forall x, P(x)$ **form:** r: $\forall x \in \mathbb{R}, x^2 \geq 0$.
- **Negation:** ~r: $\sim (\forall x \in \mathbb{R}, x^2 \geq 0) \equiv \exists x \in \mathbb{R}, \sim (x^2 \geq 0) \equiv \exists x \in \mathbb{R}, x^2 < 0$.
- **Negation in words:** ~r: There exists a real number whose square is negative.

(ii) Let **r: Student who is intelligent will succeed**. Write " r " in predicate logic and English logic, and then give its negation in both cases.

Solution.

Let P: Student;
Q: intelligent;
S: Succeed.

- **Predicate Logic:** r**:**$\forall x((\mathrm{P}(x) \wedge \mathrm{Q}(x)) \rightarrow \mathrm{S}(x))$
- **Negation:** ~r: $\sim\left[\forall x \left(\left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \rightarrow \mathrm{S}(x)\right)\right]$
  $\equiv \sim\left[\forall x \left(\sim \left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \vee \mathrm{S}(x)\right)\right]$Implication Low.
  $\equiv \exists x \left(\left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \wedge \sim \mathrm{S}(x)\right)$ De Mover's Law.
- **English logic:** ~r: There exist student who is intelligent and not succeed.

(iii) r: Some integer numbers are even but not odd.

Let $\mathbb{Z}$ ≔ **Set of Integers**, f: =**is even**, P: =**is odd**.

- **Predicate Logic:** r :$\exists x \in \mathbb{Z}, \left(f(x) \wedge \sim P(x)\right) \equiv \sim\left[\forall x\left(f(x) \longrightarrow P(x)\right)\right]$.
- **English Logic:** r: Not all even integers are odd.
- **Negation:** ~r: $\sim\sim\left[\forall x\left(f(x) \longrightarrow Q(x)\right)\right] = \left[\forall x\left(f(x) \longrightarrow Q(x)\right)\right]$.
- **Negation in words:** All even integer numbers are odd.

Remark 1.6.15.
Sometimes the English sentences are **unclear** with respect to quantification, or in another wards, quantified statements are often misused in **casual (informal) conversation**.

For example:
(i) "If you can solve any problem we come up with, then you get an A for the course"
The phrase **"you can solve any problem we can come up with"** could reasonably be interpreted as either a universal or existential quantification:

(a) “you can solve every problem we come up with”,

(b) “you can solve at least one problem we come up with”.

(ii) r**:** All students do not pay full tuition.
Here “r” could reasonably be interpreted as
(a) Not all students pay full tuition (Or: There exist some students do not pay full tuition).

(b) No students are pay full tuition (Or: There are no students are pay full tuition).
Mathematical context: Not all students pay full tuition.

(iii) r: All integer numbers are not even.”
(a) Not all integer numbers are even.
(b) No integer numbers are even (Or: There are no even integers).
Mathematical context: Not all integer numbers are even.

Combined Quantifiers 1.6.16. There are six ways in which the quantifiers can be combined when two variables are present:

(1)$\forall\ x\ \forall\ y, f(x,y) \equiv\ \forall\ y\ \forall\ x, f(x,y)$=For every x, for every $y, f(x,y)$.

(2)$\forall\ x\ \exists\ y, f(x,y)$ ≡For every x, there exists a y such that $f(x,y)$.

(3)$\forall\ y\ \exists\ x, f(x,y)$ ≡For every y, there exists an x such that $f(x,y)$.

(4)$\exists\ x\ \forall\ y, f(x,y)$ ≡There exists an x such that for every y, $f(x,y)$.

(5)$\exists\ y\ \forall\ x\ f(x,y)$ ≡There exists a y such that for every x, $f(x,y)$.

(6)$\exists\ x\ \exists\ y, f(x,y) \equiv\ \exists\ y\ \exists\ x, f(x,y)$= There exists an x such that there exists a y, $f(x,y)$.

Example 1.6.17.

(i) r:$\exists x \in \mathbb{R}\ \exists y \in \mathbb{R} : P(x,y) := (x^2 + y^2 = 2xy)$. The proposition “ r “ is true since $x = y = 1$is one of many solutions.

(ii) s:$\forall\ x \in \mathbb{R}\ \exists\ y \in \mathbb{R} : P(x,y) := (y^3 = x)$. The proposition “ s “ is true since$y = \sqrt[3]{x}$ is solution for $P(x,y)$.

(iii) s:$\exists\ x \in \mathbb{R}\ \ \forall y \in \mathbb{R} : P(x,y) := (y^3 = x)$. Here "s" mean there is an "x" real such that for every "y" real, $P(x,y)$ is true. The proposition " s " is not true since no real numbers have this property.

(iv) r: For all x, there exists y such that $xy = 1$.

Solution.

- $\forall \boldsymbol{x}, \boldsymbol{P}(\boldsymbol{x})$ **form:** r:$\forall\ x, \exists\ y$ such that $xy = 1$.
- **Negation:**~r: ~ $(\forall\ x, \exists\ y$ such that $xy = 1)$
  $\equiv\ \exists x, \sim (\exists\ y$ such that $(xy = 1))$

$\equiv\ \exists x, \forall\ y$ such that $xy \neq 1$.

- **Negation in words:** ~r: There exists $x$such that for all y, $xy \neq 1$.

(v) The following are equivalents.

(a)$\sim[\forall\ x\ \forall\ y, f(x,y)] \equiv \exists x\ \exists y, \sim f(x,y)$.

(b)$\sim[\exists\ x\ \exists y, f(x,y)] \equiv \forall x\ \forall y, \sim f(x,y)$.

(c) $\sim[\forall\ x\ \exists\ y, f(x,y)] \equiv \exists x\ \forall y, \sim f(x,y)$.

(d) $\sim[\exists\ x\ \forall\ y, f(x,y)] \equiv \forall x\ \exists y, \sim f(x,y)$.

Solution. **Exercise.**

1.7. Logical Reasoning

Definition 1.7.1. (Arguments)
An **argument** is a series of statements starting from hypothesis (premises/assumptions) and ending with the conclusion.

From the definition, an argument might be valid or invalid.

Definition 1.7.2. (Valid Arguments)(Proofs)
An argument is said to be **valid** if the hypothesis implies the conclusion; that is, if s is a statement implies from the statements $s_1, s_2, \dots, s_n$, then write as

$$s_1, s_2, \dots, s_n \mapsto s.$$

Note 1.7.3. In mathematics, the word ***proof*** is used to mean simply a valid argument. Many proofs involve more than two premises and a conclusion.

Example 1.7.4.
(i) Let s_1: Some mathematicians are engineering;
s_2 : Ali is mathematician;
s : Ali is engineering.
Show that the argument is valid.
Solution.
The argument $s_1, s_2 \mapsto s$ is not valid, since not all mathematicians are engineering.

(ii) Let s_1: There is no lazy student
s_2 : Ali is artist
s_3 : All artist are lazy
Find a conclusion s for the above premises making the argument $s_1, s_2, s_3 \mapsto s$ is valid.
Solution.
Ali is----------------------.

Remark 1.7.5.
(i) An argument $\boxed{s_1, s_2, \dots, s_n \mapsto s}$
is valid if and only if $\boxed{(s_1 \wedge s_2 \wedge \dots \wedge s_n) \to s}$ is tautology; that is,

$$\boxed{(s_1 \wedge s_2 \wedge \dots \wedge s_n) \Longrightarrow s.}$$

(ii) An argument does not depend on the truth of the premises or the conclusion but it just interested only in the question

"Is the conclusion implied by the conjunction of the premises?"

Example 1.7.6. (Example 1.4.2(i)) Show that the following argument is valid using truth table.

A_1: $p\wedge q$

A_2: $p\rightarrow\sim(q\wedge r)$

A_3: $s\rightarrow r$

C: $\therefore\sim s$

				A_1			A_2	A_3
p	**q**	**r**	**s**	**p∧q**	**(q∧r)**	**~(q∧r)=I**	**p→ I**	**s→ r**
T	T	T	T	T	T	F	F	T
T	T	T	F	T	T	F	F	T
T	T	F	T	T	F	T	T	F
T	T	F	F	T	F	T	T	T
T	F	T	T	F	F	T	T	T
T	F	T	F	F	F	T	T	T
T	F	F	T	F	F	T	T	F
T	F	F	F	F	F	T	T	T
F	T	T	T	F	T	F	T	T
F	T	T	F	F	T	F	T	T
F	T	F	T	F	F	T	T	F
F	T	F	F	F	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	T	F	F	F	T	T	T
F	F	F	T	F	F	T	T	F
F	F	F	F	F	F	T	T	T

	A_1	A_2	A_3	$A_1 \wedge A_2 \wedge A_3$	C	$(A_1 \wedge A_2 \wedge A_3) \to$ **C**
s	**p∧q**	**p→ I**	**s→ r**		∼s	
T	T	F	T	F	F	T
F	T	F	T	F	T	T
T	T	T	F	F	F	T
F	**T**	**T**	**T**	**T**	**T**	**T** ⟵
T	F	T	T	F	F	T
F	F	T	T	F	T	T
T	F	T	F	F	F	T
F	F	T	T	F	T	T
T	F	T	T	F	F	T
F	F	T	T	F	T	T
T	F	T	F	F	F	T
F	F	T	T	F	T	T
T	F	T	T	F	F	T
F	F	T	T	F	T	T
T	F	T	F	F	F	T
F	F	T	T	F	T	T

1.8. Mathematical Proof

In this section some common procedures of proofs in mathematics are given with examples.

To Prove Statement of Type (p → q) 1.8.1.

(1) Rule of conditional proof.

Let p is true statement and $s_1, s_2, \dots, s_n$ all previous axioms and theorems. To prove p → q it is enough to prove

$$s_1, s_2, \dots, s_n, \text{p} \mapsto \text{q}$$

is valid argument.

Example 1.8.2. Prove that, a is an even number → a^2 is an even number.

Proof.

Suppose a is an even number.

(1) $a = 2k, k$ is an integer (definition of even number).

(2) $a^2 = 4k^2$, square both sides of (1)

(3) $a^2 = 2(2k^2)$, Common factor

(4) a^2 is even number, since $2k^2$ is an integer and definition of even number.

Note that in the above proof, we proved the tautology

$$(s_1 \wedge s_2 \wedge \text{p}) \to \text{q},$$

where

p: a is an even number

s_1: $a = 2k$,

s_2: $a^2 = 4k^2$,

q: a^2 is even number.

(2) Contrapositive

To prove p → q we can proof that (~ q → ~p) since (p → q) ≡ (~ q → ~p).

Example 1.8.3. Prove that, (a^2 is an even number) → (a is an even number).

Proof.

Let p: $a^2$is an even number,

q: a is an even number.

Then

~p: $a^2$is an odd number,

~q: $a$is an odd number.

Therefore, the contrapositive statement is

$$\boxed{a \text{ is an odd number } \to a^2 \text{ is an odd number}}.$$

(1) $a = 2k + 1$ k is an integer (Definition of odd number)
(2) $a^2 = 4k^2 + 4k + 1$ Square both sides of (1)
(3) $a^2 = 2(2k^2 + 2k) + 1$
(4) a^2 is odd number since $2k^2 + 2k$ is an integer and definition of odd number.

Prove Statement of Type (p ⟷ q) 1.8.4.
(i) Since (p ⟶ q) ∧ (q ⟶ p) ≡ (p ⟷ q), so we can proved first p ⟶ q and then proved q ⟶ p.
(ii) Moved from p into q through series of logical equivalent statements s_i as follows:

$$\begin{gathered} p \leftrightarrow s_1 \\ s_1 \leftrightarrow s_2 \\ \vdots \\ s_{n-1} \leftrightarrow s_n \\ s_n \leftrightarrow q \end{gathered}$$

This is exactly the tautology

$$((p \leftrightarrow s_1) \wedge (s_1 \leftrightarrow s_2) \wedge \ldots \wedge (s_n \leftrightarrow q)) \rightarrow (p \leftrightarrow q).$$

Prove Statement of Type $\forall \boldsymbol{x}\, \boldsymbol{P}(\boldsymbol{x})$ or $\exists \boldsymbol{x}\, \boldsymbol{P}(\boldsymbol{x})$ 1.8.5.

(i) To prove a sentence of type $\forall\, x\, P(x)$, we suppose x is an arbitrary element and then prove that $P(x)$ is true.
(ii) To prove a sentence of type $\exists\, x\, P(x)$, we have to prove there exist at least one element x such that $P(x)$ is true.

Prove Statement of Type (p∨r)⟶ q 1.8.6.
Depending on the tautology

$$[(p \rightarrow q) \wedge (r \rightarrow q)] \rightarrow [(p \vee r) \rightarrow q]$$

We must prove that p⟶q and r⟶q.

Example 1.8.7. Prove that $(a = 0 \vee b = 0) \rightarrow (ab = 0)$ where a, b are real numbers.
Proof.
Firstly, we prove that $(a = 0) \rightarrow (ab = 0)$.
Suppose that $a = 0$, then $ab = 0. b = 0$.
Secondly, we prove that $(b = 0) \rightarrow (ab = 0)$.
Suppose that $b = 0$, then $ab = a. 0 = 0$.
Therefore, the statement $(a = 0 \vee b = 0) \rightarrow (ab = 0)$ is tautology.

Proof by Contradiction 1.8.8.
The contradiction is always false statement whatever the truth values of its components. Proof by contradiction is type of indirect proof.

The way of proof logical proposition **p** by contradiction start by supposing that **∼p** and then try to find sentence of type

$$R \wedge \sim R$$

where R is any sentence contain **p** or any pervious theorem or any axioms or any logical propositions.
By this way we can also prove sentences of type $\boxed{\forall x\, P(x)}$ or $\boxed{\exists x\, P(x)}$ or $\boxed{(\mathrm{p} \rightarrow \mathrm{q})}$ or $\boxed{(\mathrm{p} \Longrightarrow \mathrm{q})}$.

Example 1.8.9. Prove that $(x \neq 0) \Longrightarrow (x^{-1} \neq 0)$, x is real number.
Proof.
Let p: $x \neq 0$,
q: $x^{-1} \neq 0$.
We must prove p ⟹q.
Suppose ∼(p ⟹q) is true.
(1) ∼ (p → q) is tautology Def. of logical implication.
(2) ∼(p → q)≡ ∼ (∼ p⋁q)Implication Law
(3) p ∧ ∼q is tautology, De Morgan's Law
(4) $x \neq 0 \wedge x^{-1} = 0$.
(5) $x \cdot x^{-1} = 1 \neq 0$.
(6) $x \cdot x^{-1} = x \cdot 0 = 0$.
(7) $1 = 0$, Inf. (5), (6).
(8) This is contradiction, since $(1 \neq 0) \wedge (1 = 0)$ Contradiction Law
Thus, the statement ∼(p ⟹q) is not true. Therefore, p⟹q.

Application Example:
Cryptography (التشفير)

A	**1 0 1 0 0**	**Q**	**1 0 0 1 1**
B	**0 0 0 1 0**	**R**	**1 0 0 1 0**
C	**1 0 1 0 1**	**S**	**1 0 0 0 0**
D	**0 0 1 1 0**	**T**	**0 1 1 1 0**
E	**1 0 1 1 0**	**U**	**0 0 0 1 1**
F	**1 0 1 1 1**	**V**	**0 1 1 0 1**
G	**1 1 0 0 0**	**W**	**0 1 1 1 1**
H	**1 1 0 1 0**	**X**	**0 0 1 0 0**
I	**0 0 0 0 1**	**Y**	**0 1 1 0 0**
J	**1 1 0 0 1**	**Z**	**1 0 0 0 1**
K	**0 0 1 1 1**	**Space**	**1 1 1 1 1**
L	**0 1 0 1 1**	**0**	**1 1 0 1 1**
M	**0 1 0 1 0**	**1**	**1 1 1 0 0**
N	**0 1 0 0 1**	**2**	**1 1 1 0 1**
O	**0 1 0 0 0**	**3**	**1 1 1 1 0**
P	**0 0 1 0 1**	**4**	**0 0 0 0 0**

Key: 0 0 1 0 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1 0 0 0

Plaintext: GO HOME

Plaintext	**G**	**O**		**H**	**O**	**M**	**E**
Code	**11000**	**01000**	**11111**	**11010**	**01000**	**01010**	**10110**
Key	**00101**	**01100**	**11010101111000 00101011001**				
XOR							
Encryption	**11101**	**00100**	**00101**	**01101**	**11000**	**00000**	**01111**
Ciphertext	**2**	**X**	**P**	**V**	**G**	**4**	**W**

Ciphertext	**2**	**X**	**P**	**V**	**G**	**4**	**W**
Code	**11101**	**00100**	**00101**	**01101**	**11000**	**00000**	**01111**
Key	**00101**	**01100**	**11010**	**10111**	**1000001010**		**11001**
XOR							
Decryption	**11000**	**01000**	**11111**	**11010**	**01000**	**01010**	**10110**
Plaintext	**GOHOME**						

Exercise

Q1: Show that

(1) $(p \to q) \wedge \sim q \Rightarrow \sim p$.
(2) $p \wedge (p \to q) \to \sim q$ is a contingency using a truth table.
(3) $p \to (p \vee q)$ is a tautology using a truth table.
(4) $(p \wedge q) \to p$ is a tautology using a truth table and logical equivalences.
(5) $(p \wedge q) \to (p \vee q)$ is a tautology using a truth table and logical equivalences.
(6) $[p \to (q \to r)] \equiv [(p \wedge q) \to r]$ using a truth table and logical proposition.
(7) $[p \to (q \to r)] \equiv [q \to (p \to r)]$ using a truth table and logical proposition.
(8) $[(p \wedge q) \to p] \equiv [q \to (p \vee \sim p)]$ using a truth table and logical proposition.
(9) $[(p \to q) \wedge (p \to r)] \equiv [p \to q \wedge r)]$ using a truth table and logical proposition.

(10) $[(p \to q) \wedge (r \to q)] \equiv [(p \vee r) \to q]$ using a truth table and logical proposition.

(11) $(p \to q) \equiv (\sim q \to \sim p)$ using a truth table and logical proposition.

(12) $p \wedge (\sim p \vee q) \equiv p \wedge q$ using a truth table and logical proposition.
(13) $p \vee (p \wedge q) \equiv p$ using a truth table and logical proposition.

(14) Is $\veebar$ commutative or associative?

(15) Is $\veebar$ distributive over $\wedge$, $\vee$, or $\to$?

(16) Is this true $p \veebar q \equiv p \leftrightarrow \sim q$?

(17) $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$ using a truth table.
(18) $[(p \vee q) \wedge \sim p]) \Rightarrow q$ using a truth table.
(19) $[(p \to q) \wedge (r \to s)] \Rightarrow [(p \vee r) \to (q \vee s)]$ using a truth table.

(20) $[(p \to q) \wedge (p \vee r)] \Rightarrow (q \vee r)$ using a truth table.

Q2: Given the hypotheses:

(i) "It is not sunny this afternoon and it is colder than yesterday"
(ii) "We will go swimming only if it is sunny"
(iii) "If we do not go swimming, then we will take a canoe trip"
(iv) "If we take a canoe trip, then we will be home by sunset"
Does this imply that **"we will be home by sunset"?**

Q3: Represent as propositional expressions, and use De Morgan's Laws to write the negation of the expression, and translate the negation in English.

"Tom is a math major but not computer science major."

Q4: Let

p = John is healthy;
q = John is wealthy;
r = John is wise.

Represent symbolically:

(i) John is healthy and wealthy but not wise.

(ii) John is not wealthy but he is healthy and wise.

(iii) John is neither healthy nor wealthy nor wise.

Q5: Translate the sentences into propositional expressions:

"Neither the fox nor the lynx can catch the hare if the hare is alert and quick."

Q6: Represent as propositional expressions.

"You can either (stay at the hotel and watch TV) or (you can go to the museum and spend some time there)".

Q7: Given a sentence **"If we are on vacation, we go fishing."** Then

(i) translate the sentence into a logical expression,

(ii) write the negation of the logical expression and translate the negation into English,

(iii) write the converse of the logical expression and translate the converse into English,

(iv) write the inverse of the logical expression and translate the inverse into English,

(v) write the contrapositive of the logical expression and translate the contrapositive into English.

Q8: Write the contrapositive, converse and inverse of the expressions:

$$p \rightarrow q;$$

$$\sim p \rightarrow q;$$

$$q \rightarrow \sim p.$$

Q9: Determine whether the following arguments are valid or invalid:

(i) Premises:

(a) If I read the newspaper in the kitchen, my glasses would be on the kitchen table.

(b) I did not read the newspaper in the kitchen.

Conclusion: My glasses are not on the kitchen table.

(ii) Premises:

(a) If I don't study hard, I will not pass this course

(b) If I don't pass this course I cannot graduate this year.

Conclusion: If I don't study hard, I won't graduate this year.

(iii) Premises:

(a) You will get an extra credit if you write a paper or if you solve the test problems.

(b) You don't write a paper, however you get an extra credit.

Conclusion: You have solved the test problems.

Q10: Find an expression equivalent to $p \rightarrow q$ that uses only $\wedge$ and $\sim$.

Q11: Negate the following sentences.

(i) The number x is positive, but the number y is not positive.
(ii) If x is prime, then $\sqrt{x}$ is not a rational number.
(iii) For every prime number p, there is another prime number q with $q > p$.

(iv) There exists a real number a for which $a + x = x$ for every real number x.
(v) Every even integer greater than 2 is the sum of two primes.
(vi) The integer x is even, but the integer y is odd.
(vii) At least one of the integers x and y is even.
(viii) The numbers x and y are both odd.
(ix) For every real number x there is a real number y for which $y^3 = x$.
(x) I don't eat anything that has a face.

Q12: Write the following propositions with quantifiers and then give its negation with translations into words.

(i) Some counting numbers are greater than five

(ii) Every element of set D is less than 7.

(iii) Some elements of set D are less than 13.

(iv) Every counting number greater than 4 is greater than 2.
(v) Some counting numbers are even.
(vi) Every counting number which is divisible by 2 is even.
(vii) Every counting number is even or odd.

(viii) For every x in D_x and for every $y \in$ in D_x, x plus y less than 3.

(ix) At least one politician isn't a logician.

(x) Only no logicians are politicians.

Q13: Prove that $\sqrt{2}$ is irrational using contradiction method.